Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 4, Number 3 (2012), pp. 249-255

http://www.irphouse.com

Generalization of Quasi-Differentiable Maps

Sushil Kumar

Applied Science Department, Bharati Vidhyapeeth's College of Engineering, New Delhi-110063, India
E-mail: sushilkumar16n@gmail.com

Abstract

The concept of quasi differentiability was introduced in 1995 by A.Bayoumi and this Differentiability is stronger than Frechet differentiability. In this paper, a new concept of differentiability 'Weak Quasi Differentiable Maps' has been introduced and it's some characterizations like linearity, Lipschitzian property chain rule and etc. have been derived.

2000 Mathematics Subject Classification: 46A16, 46E50

Keywords: Quasi differentiable map, Chain rule, Frechet Differentiable map

Introduction

Differentiability of a function on normed spaces is the most important concept in analysis. Frechet and Gateaux differentiability of a function on normed spaces some kind of differentiability [1, 2, 3].

In 1995, A. Bayoumi [4, 5] introduced a concept of differentiability, is known as quasi-differentiability in *F*-spaces.

Let E and F be p-normed space and q-normed space respectively $(0 < p, q \leq 1)$ and U , open subset of E.

A mapping $f : U \to F$ is said to be *m*-quasi differentiable or $m\, pq$ differentiable at $a \in U$, if there exists a linear map $T_a \in L(E, F)$, such that

$$\lim_{x \to a} \frac{\| f(x) - f(a) - T_a(x-a) \|^{\frac{m}{q}}}{\| x - a \|^{\frac{1}{p}}} = 0 \tag{1.1}$$

T_a is called pq -differential or quasi differential of function f at point a

If $m = 1$

Then this mapping is known as super differentiable mapping at a.

i.e. for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$0 < \|x - a\| < \delta$

$$\Rightarrow \qquad \| f(x) - f(a) - T_a(x-a) \| \le \varepsilon \| x - a \|^{\frac{q}{p}}$$

$$\text{i.e.} \qquad \| f(a+h) - f(a) - T_a(h) \| \le \varepsilon \| h \|^{\frac{q}{p}} \tag{1.2}$$

Furthermore, A. Bayoumi [6, 7] discussed the various properties of pq-differentiable mappings. In 2006, A. Bayoumi [8] shows that Quasi differentiability of maps may not be Frechet differentiability of maps by some examples. The definition of weak holomorphic map [9] is

Let E and F be two complex normed space, and let U be open subset of E.

A mapping $f : U \to F$ is said to be weak holomorphic on U if

$\lambda \circ f : U \to \boldsymbol{C}$ is holomorphic for every $\lambda \in F^*$

In this note the weak quasi differentiable map between locally bounded spaces and some characterizations of this map have been derived.

In section 2 the concept of 'Weak quasi tangent' and 'weak quasi differentiable maps is introduced. In section 3 some properties of this map is derived, and at last in section 4 a result related mapping into product space is derived.

Weak quasi differentiable maps

Let E and F be a p-normed space and q-normed space respectively $(0 < p, q \le 1)$ and U, be open in E.

2.1 Mappings f,g: U→F are weak quasi tangent to each other if $(\psi \circ f)$ and $(\psi \circ g)$ are quasi tangent to each other at a

$$\text{i.e.} \qquad \lim_{x \to a} \frac{\|(\psi \circ f)(x) - (\psi \circ g)(x)\|^{\frac{1}{q}}}{\|x - a\|^{\frac{1}{p}}} = 0, \quad \forall \psi \in L(F) \tag{2.1}$$

2.2 A mapping: $f : U \to F$ is said to be weak m-quasi differentiable map if $\psi \circ f$ is m-quasi differentiable map for every $\psi \in F^*$.

i.e. at $a \in U$, if $\exists$ a continuous linear mapping $\psi \circ T$ at a $(T \in L(E, F)$ and by the definition of quasi differentiable map) such that

$$\lim_{x \to a} \frac{\| (\psi \circ f)(x) - (\psi \circ f)(a) - (\psi \circ T)_a (x-a) \|^{\frac{m}{q}}}{\| x - a \|^{\frac{1}{p}}} = 0 \tag{2.2}$$

If $m=1$, f is called weak quasi differentiable at a.

Hence, for every $\in > 0, \exists \delta > 0 : 0 < \| x-a \| < \delta$.

$$\Rightarrow \quad \| (\psi \circ f)(x) - (\psi \circ f)(a) - (\psi \circ T)_a (x-a) \| \leq \varepsilon \| x-a \|^{\frac{q}{p}} \tag{2.3}$$

If f is weak quasi differentiable at each point of U, then f is said to be weak quasi differentiable on U.

Properties of weak quasi differentiable map

Theorem 3.1: (Linearity) The set of weak quasi differentiable mappings at $a \in U$ form a vector space.

Proof: Let S be the set of weak quasi differentiable mappings.
If $f, g \in S$, then by definition of weak quasi differentiable map, there exist linear mappings

$(\psi \circ T)_a$ and $(\psi \circ T^*)_a \in L(F)$ for all $\psi \in F^*$

such that

$$\| (\psi \circ f)(x) - (\psi \circ f)(a) - (\psi \circ T)_a (x-a) \|^{\frac{1}{q}} \leq \frac{\varepsilon}{2} \| x-a \|^{\frac{1}{p}} \tag{3.1}$$

$$\| (\psi \circ g)(x) - (\psi \circ g)(a) - (\psi \circ T^*)_a (x-a) \|^{\frac{1}{q}} \leq \frac{\varepsilon}{2} \| x-a \|^{\frac{1}{p}} \tag{3.2}$$

For a given $\varepsilon > 0,$ since $\| \cdot \|^{\frac{1}{q}}$ is a weak quasi norm.

$$\| (\psi \circ f + \psi \circ g)(x) - (\psi \circ f + \psi \circ g)(a) - (\psi \circ T + \psi \circ T^*)_a (x-a) \|^{\frac{1}{q}}$$

$$\leq \sigma(\| (\psi \circ f)(x) - (\psi \circ f)(a) - (\psi \circ T)_a (x-a) \|^{\frac{1}{q}} + \| (\psi \circ g)(x) - (\psi \circ g)(a) - (\psi \circ T^*)_a (x-a) \|^{\frac{1}{q}})$$

Dividing on both sides by $\| x-a \|^{\frac{1}{p}}$ and taking limit $x \to a$

$$\lim_{x \to a} \frac{\| (\psi \circ f + \psi \circ g)(x) - (\psi \circ f + \psi \circ g)(a) - (\psi \circ T + \psi \circ T^*)_a (x-a) \|^{\frac{1}{q}}}{\| x-a \|^{\frac{1}{p}}}$$

$$\leq \sigma[\lim_{x \to a} \frac{\| (\psi \circ f)(x) - (\psi \circ f)(a) - (\psi \circ T)_a (x-a) \|^{\frac{1}{q}}}{\| x-a \|^{\frac{1}{p}}}$$

$$+ \lim_{x \to a} \frac{\| (\psi \circ g)(x) - (\psi \circ g)(a) - (\psi \circ T^*)_a (x-a) \|^{\frac{1}{q}}}{\| x-a \|^{\frac{1}{p}}}]$$

$$\Rightarrow \qquad D(\psi \circ f + \psi \circ g)(a) = D(\psi \circ f)(a) + D(\psi \circ g)(a) \tag{3.3}$$

Let $\lambda \neq 0$ be a scalar

Now, we have to prove

$D(\lambda(\psi \circ f)(a)) = \lambda D(\psi \circ f)(a)$

$\lambda \neq 0$

For $D(\lambda \cdot \psi o f)(a)$

$$\lim_{x \to a} \frac{\| \lambda[(\psi \circ f)(x) - (\psi \circ f)(a) - (\psi \circ T)_a (x-a)] \|^{\frac{1}{q}}}{\| x-a \|^{\frac{1}{p}}}$$

$$= \lim_{x \to a} \frac{\| \lambda(\psi \circ f)(x) - \lambda(\psi \circ f)(a) - \lambda(\psi \circ T)_a (x-a) \|^{\frac{1}{q}}}{\| x-a \|^{\frac{1}{p}}}$$

$$\Rightarrow \qquad D(\lambda \cdot (\psi \circ f)(a) = \lambda D(\psi \circ f(a)) \tag{3.4}$$

Theorem 3.2: (Lipschitzian property) Let E and F be p-normed and q-normed spaces respectively, $(0 < p, q \leq 1)$ and U, open in E.

If $f: U \to F$ is weak quasi differentiable map at $a \in U$, then there exist $c > 0$ and $\delta > 0$ such that

$$\| (\psi \circ f)(x) - (\psi \circ f)(a) \| \leq c \| x-a \|^{\frac{q}{p}}$$

for $x \in U$, $\| x-a \| < \delta$.

Proof: Let $T = D(\psi \circ f)(a)$

$$\Delta(x) = (\psi \circ f)(x) - (\psi \circ f)(a) - T(x-a) \text{ for } x \in U \tag{3.5}$$

Then

$\| (\psi \circ f)(x) - (\psi \circ f)(a) \| = \| T(x-a) + \Delta(x) \|$

$\leq \| T \|^q \| x-a \|^{\frac{q}{p}} + \| \Delta x \| \quad (\because \quad \| T(x-a) \| \leq \| T \|^q \| x-a \|^{\frac{q}{p}})$

Since

$\lim_{x \to a} \frac{\| \Delta(x) \|^p}{\| x-a \|^q} = 0$ and $\Delta(a) = 0$

Given $\in = \frac{1}{2}, \exists \delta > 0$ such that

$\| x-a \| \leq \delta,\ x \in U,$ then

$$\|\Delta(x)\| \le \frac{1}{2}\|x-a\|^{\frac{q}{p}}$$

Therefore

$$\|(\psi\circ f)(x)-(\psi\circ f)(a)\| \le (\|T\|^{q}+\frac{1}{2})\|x-a\|^{\frac{q}{p}}$$

$$\Rightarrow \ \|(\psi\circ f)(x)-(\psi\circ f)(a)\| \le c\|x-a\|^{\frac{q}{p}} \tag{3.6}$$

where $c=\|T\|^{q}+\frac{1}{2}$.

Theorem 3.3: (Chain Rule). Let $f:U\to F$ be weak quasi differentiable at $a\in U$ and let $T\in L(F,G)$ where E,F are p-normed space and q-normed space and U is open in E, then $(T\circ f)$ is weak quasi differentiable at $a\in U$.

Proof: We have to prove,

$(T\circ f)$ is weak quasi differentiable at a

$\Rightarrow$ $\psi\circ(T\circ f)$ is quasi differentiable at a.

Let $S=T\circ f$

Therefore

$$\|(\psi\circ S)(x)-(\psi\circ S)(a)-\psi\circ D(S(x-a)\|^{\frac{1}{q}} = \|\psi(S(x)-S(a)-DS(x-a)\|^{\frac{1}{q}}$$

$$\le \|\psi\|^{\frac{1}{q}}\ \|S(x)-S(a)-D(S)(x-a)\|^{\frac{1}{q}}$$

Dividing on both sides by $\|x-a\|^{\frac{1}{p}}$ and
Taking the $\lim x\to a$, we get

$$\frac{\|(\psi\circ S)(x)-(\psi\circ S)(a)-\psi\circ DS(x-a)\|^{\frac{1}{q}}}{\|x-a\|^{\frac{1}{p}}} \le \frac{\|\psi\|^{\frac{1}{q}}\|S(x)-S(a)-DS(x-a)\|^{\frac{1}{q}}}{\|x-a\|^{\frac{1}{p}}}$$

$$=\frac{\|\psi\|^{\frac{1}{q}}\|(T\circ f)(x)-(T\circ f)(a)-D(T\circ f)(x-a)\|^{\frac{1}{q}}}{\|x-a\|^{\frac{1}{p}}}$$

$$=\frac{\|\psi\|^{\frac{1}{q}}\|(T\circ f)(x)-(T\circ f)(a)-(T\circ Df)(x-a)\|^{\frac{1}{q}}}{\|x-a\|^{\frac{1}{p}}}$$

$$=\frac{\|\psi\|^{\frac{1}{q}}\|T\|^{\frac{1}{q}}\|\ f(x)-f(a)-Df(x-a)\|^{\frac{1}{q}}}{\|x-a\|^{\frac{1}{p}}} \qquad \to 0$$

$\Rightarrow \quad T \circ f$ is weak quasi differentiable.

Mapping into product space

Theorem 4: Let $U \subset E$ be open in a p-normed space E and F_n-normed spaces $(n = 1, 2 \ldots m)$. A mapping $f : U \to F_1 \times F_2 \times F_3 \cdots F_n$ is weak quasi differentiable at $a \in U$ if and only if each coordinate map $f_n = \pi_n \circ f$ is weak quasi differentiable at a

Further $Df(a) = (Df_1(a), Df_2(a) \ldots Df_m(a))$

where π_n is the projection from $F_1 \times F_2 \times F_3 \ldots \times F_m$ onto F_n.

Proof: Let F_n be p_n-normed spaces for $n = 1, 2 \ldots m$.

And let $F = F_1 \times F_2 \times F_3 \times \ldots \times F_n$

Then F is an F-space and topology induced by an F-norm on F, is the product topology.

Let $\pi_n : F \to F_n$ be the projection mappings onto the n^{th} factor F_n and let $u_n : F_n \to F$ be natural embedding map defined by

$u_n(x_n) = (0, 0, \ldots 0, x_n, 0, 0 \ldots 0)$

Then both π_n and u_n are continuous linear maps.

$\pi_n o u_n = 1_{F_n}$ (the identify map on F_n)

$\sum u_n o \pi_n = 1_F$ (the identify map on F)

Let U be an open subset of E and let $f : U \to F$ and let

$f_n = \pi_n o f : U \to F_n$ be the n^{th} coordinate map. Then

$$f = \sum_{n=1}^{m} u_\eta \circ \pi_n o f = \sum_{n=1}^{m} u_\eta \circ f_\eta = (f_1, f_2 \cdots f_m)$$

So, if f is weak quasi differentiable at a, then $u_\eta \circ \pi_n$ is quasi differentiable at a, then by chain rule (Theorem 3.3)

$f(a) = \sum u_n \circ Df_n(a)$

$= (Df_1(a), Df_2(a) \ldots Df_m(a))$

Conversely

If each f_n is weak quasi differentiable at a then f is clearly weak quasi differentiable at a.

References

[1] H.Cartan, 1971, Differential Calculus, Hermann, Paris.
[2] Walter Rudin, 1973, Functional Analysis, McGrahil.
[3] J Muzica, 1986, Complex Analysis in Banach Spaces, North Holland Math. Studies.
[4] A. Bayoumi, 2003, Foundations of complex analysis in Non locally convex spaces, North Holland Publication, Mathematics Studies.
[5] A. Bayoumi, 2006, "Super differential calculus in F -spaces", Research report in mathematics, no 7.
[6] A. Bayoumi, 1997, "Mean value theorem for complex bounded locally space", Communication in Applied Non-linear Analysis (4) , pp. 91-103.
[7] A. Bayoumi, 2005, "Bolzano's Intermediate-value theorem for quasi-holomorphic maps", Central European Journal of Mathematics, 3(1), pp. 76-82.
[8] Soo Bong Chae, Holomorphy and calculus in Normed spaces, Marcel Dekker, Inc., New York.
[9] A. Bayoumi, 2006, "Bayoumi differential map is different from Frechet Differential map", Central European Journal of Mathematics, 4(4) , pp., 585-593.