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The Hermitian Positive Definite Solutions of the Matrix Equation $X^s + A^* x^{-n} A = P$

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In this paper, the Hermitian positive definite solutions of the matrix equation $X^s + A^* X^{-n} A = P$ is considered, where A is an $m \times m$ nonsingular matrix and P is an $m \times m$ Hermite positive definite matrix, s is a positive real number, n *is a* natural number. Necessary and sufficient conditions for the existence of an Hermitian positive definite solution are derived, and a iterative solution is provided.

Keywords : Matrix Equation, Hermitian positive Definite Solution, iterative Solution

Introduction

We consider the matrix equation

$$X^s + A^* X^{-n} A = P \tag{1.1}$$

where A is an $m \times m$ nonsingular matrix and P is an $m \times m$ Hermite positive definite matrix, s is a positive real number, n *is a* natural number. We mainly discuss the symmetric positive definite solutions of Eq. (1.1).

Solvability Conditions and Iterative Solution

We consider the following two polynomial equations

$$x^{s+n} - \lambda_{\min}(P) x^n + \lambda_{\max}(A^* A) = 0 \tag{2.1}$$

and

$$x^{s+n} - \lambda_{\max}(P) x^n + \lambda_{\min}(A^* A) = 0 \tag{2.2}$$

Let $g(x)=\lambda_{\min}^{s}(P)x^{n}-x^{s+n}$, $\xi_{*}=\lambda_{\min}^{\frac{1}{s}}(P)(\frac{n}{n+s})^{\frac{1}{s}}=\left\|P^{-1}\right\|^{\frac{-1}{s}}(\frac{n}{n+s})^{\frac{1}{s}}$,

It is easy to verify that the necessary and sufficient condition for the existence of the positive real root of Eq. (2.1) and Eq. (2.2) is

$$\lambda_{\max}(A^{*}A)<\lambda_{\min}(P)\xi_{*}{}^{n}-\xi_{*}^{s+n}$$

$$=\lambda_{\min}(P)[\lambda_{\min}^{\frac{1}{s}}(P)(\frac{n}{n+s})^{\frac{1}{s}}]^{n}-\lambda_{\min}^{\frac{s+n}{s}}(P)(\frac{n}{n+s})^{\frac{s+n}{s}}$$

$$=\lambda_{\min_{n}}^{\frac{n+s}{s}}(P)[(\frac{n}{n+s})^{\frac{n}{s}}-(\frac{n}{n+s})^{\frac{s+n}{s}}]$$

$$=\lambda_{\min_{n}}^{\frac{n+s}{s}}(P)(\frac{n}{n+s})^{\frac{n}{s}}\frac{s}{n+s}$$

$$=\xi_{*}^{n+s}\frac{1}{\frac{n}{n+s}}\frac{s}{n+s}$$

$$=\xi_{*}^{n+s}\frac{s}{n}$$

Thus, in this subsection we assume that A satisfies

$$\left\|A\right\|^{2}\left\|P^{-1}\right\|^{\frac{n+s}{s}}\leq(\frac{n}{n+s})^{\frac{n}{s}}\frac{s}{n+s} \tag{2.3}$$

By (2.3) we know that Eq.(2.1) has two positive real roots $\alpha_2<\beta_1$, Eq.(2.2) has two positive real roots $\alpha_1<\beta_2$ .It is easy to prove that,

$$0<\alpha_1\leq\alpha_2<\xi_*<\beta_1\leq\beta_2$$

Theorem 2.1: Suppose that A and P satisfies(2.3), X is the solution of Eq.(1.1).Then

$$\alpha_1\leq\lambda_{\min}(X)\leq\alpha_2 \text{ or } \beta_1\leq\lambda_{\min}(X)\leq\beta_2$$

$$\alpha_1\leq\lambda_{\max}(X)\leq\alpha_2 \text{ or } \beta_1\leq\lambda_{\max}(X)\leq\beta_2 .$$

Proof: Suppose X is the solution of Eq.(1.1), then $X^{s}+A^{*}X^{-n}A=P$, by Wyle inequation we know that,

$$\lambda_{\min}^{s}(X)=\lambda_{\min}(P-A^{*}X^{-n}A)\geq\lambda_{\min}(P)-\lambda_{\max}(A^{*}X^{-n}A)\geq\lambda_{\min}(P)-\frac{\lambda_{\max}(A^{*}A)}{\lambda_{\min}^{n}(X)}.$$

$$\lambda_{\max}^{s}(X)=\lambda_{\max}(P-A^{*}X^{-n}A)\le\lambda_{\max}(P)-\lambda_{\min}(A^{*}X^{-n}A)\ge\lambda_{\max}(P)-\frac{\lambda_{\min}(A^{*}A)}{\lambda_{\max}^{n}(X)}$$

So, $\lambda_{\min}(X)\ge\beta_1$ or $\lambda_{\min}(X)\le\alpha_2$; $\alpha_1\le\lambda_{\max}(X)\le\beta_2$.

On the other hand, $X^{s}+A^{*}X^{-n}A=P$, so $X^{n}=A(P-X^{s})^{-1}A^{*}$, then

$$\frac{A^{*}A}{\lambda_{\min}(P-X^{s})}\ge A(P-X^{s})^{-1}A^{*}=X^{n}\ge\frac{AA^{*}}{\lambda_{\max}(P-X^{s})}$$

thus $\lambda_{\min}^{n}(X)\lambda_{\max}(P-X^{s})\ge\lambda_{\min}(AA^{*})$, $\lambda_{\max}^{n}(X)\lambda_{\min}(P-X^{s})\le\lambda_{\max}(AA^{*})$

by Wyle ineqution, we can obtain

$$\lambda_{\max}(P-X^{s})\le\lambda_{\max}(P)-\lambda_{\min}^{s}(X),\ \lambda_{\min}(P-X^{s})\ge\lambda_{\min}(P)-\lambda_{\max}^{s}(X)$$

so, $\lambda_{\max}(P)\lambda_{\min}^{n}(X)-\lambda_{\min}^{n+s}(X)\ge\lambda_{\min}(AA^{*})$, $\lambda_{\min}(P)\lambda_{\max}^{n}(X)-\lambda_{\max}^{n+s}(X)\le\lambda_{\max}(AA^{*})$

so, $\alpha_1\le\lambda_{\min}(X)\le\beta_2$, $\lambda_{\max}(X)\ge\beta_1$ or $\lambda_{\max}(X)\le\alpha_2$.

Remark 2.1: Suppose that A and P satisfies(2.3), X is the solution of Eq.(1.1), there are no solutions in $[\alpha_2 I,\beta_1 I]$.

Remark 2.2: Suppose that A and P satisfies(2.3), X is the solution of Eq.(1.1), then

$$X\in[\alpha_1 I,\alpha_2 I]\cup[\beta_1 I,\beta_2 I]\cup\{X^{*}=X\ |\alpha_1\le\lambda_{\min}(P)\le\alpha_2,\beta_1\le\lambda_{\max}(P)\le\beta_2\}.$$

Theorem 2.2: Suppose that A and P satisfies(2.3), We define sequences as the following,

$$\hat{X}_0=\hat{\gamma}I\in[\alpha_2 I,\beta_1 I],\ \hat{X}_{k+1}=[A(P-\hat{X}_k^{s})^{-1}A^{*}]^{\frac{1}{n}},\ k=0,1,2...$$

$$\check{X}_0=\check{\gamma}I\in[0,\alpha_1 I],\ \check{X}_{k+1}=[A(P-\check{X}_k^{s})^{-1}A^{*}]^{\frac{1}{n}},\ k=0,1,2...$$

the sequences satisfy $\check{X}_0\le\check{X}_1\le\cdots\le\check{X}_n\le\cdots\le\check{X}\le\hat{X}\le\cdots\le\hat{X}_n\le\cdots\le\hat{X}_1\le\hat{X}_0$,

then there is a maximal solution $\hat{X}$ and a minimal solution $\check{X}$,

$$\hat{X}=\lim_{n\to\infty}\hat{X}_n,\ \check{X}=\lim_{n\to\infty}\check{X}_n.$$

Proof: Let $h(X)=[A(P-X^s)^{-1}A^*]^{\frac{1}{n}}$, $\Phi=[\alpha_1 I,\alpha_2 I]$, then $\forall X\in\Phi$, $P-X^s>0$.

$$\lambda_{\min}h(X)=\lambda_{\min}^{\frac{1}{n}}[A(P-X^s)^{-1}A^*]\ge[\frac{\lambda_{\min}(AA^*)}{\lambda_{\max}(P)-\alpha_1^s}]^{\frac{1}{n}}=\alpha_1 .$$

$$\lambda_{\max}h(X)=\lambda_{\max}^{\frac{1}{n}}[A(P-X^s)^{-1}A^*]\le[\frac{\lambda_{\max}(AA^*)}{\lambda_{\min}(P)-\alpha_2^s}]^{\frac{1}{n}}=\alpha_2$$

So, $h(\Phi)\subset\Phi$.

$\forall X,Y\in\Phi, X>Y$, $h(X)=[A(P-X^s)^{-1}A^*]^{\frac{1}{n}}>[A(P-X^s)^{-1}A^*]^{\frac{1}{n}}=h(Y)$, that is to say, $h(X)$ is monotone increasing.

$$h(\hat{X}_0)=[A(P-\hat{X}_0^{\,s})^{-1}A^*]^{\frac{1}{n}}\le[\frac{\lambda_{\max}(AA^*)}{\lambda_{\min}(P)-\hat{\gamma}^s}I]^{\frac{1}{n}}\le\hat{\gamma}I=\hat{X}_0$$

$$h(\check{X}_0)=[A(P-\check{X}_0^{\,s})^{-1}A^*]^{\frac{1}{n}}\le[\frac{\lambda_{\min}(AA^*)}{\lambda_{\max}(P)-\check{\gamma}^s}I]^{\frac{1}{n}}\le\check{\gamma}I=\check{X}_0$$

It is easy to verify the sequences $\{\hat{X}_n\}$, $\{\check{X}_n\}$ are convergent,

$$\hat{X}=\lim_{n\to\infty}\hat{X}_n,\ \check{X}=\lim_{n\to\infty}\check{X}_n .$$

Collary. Suppose that A and P satisfies(2.3), X is the solution of Eq.(1.1).If $X\in[\alpha_1 I,\alpha_2 I]\}$, then

$$\check{X}\le X\le\hat{X} .$$

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