The Hermitian Positive Definite Solutions of the Matrix Equation $X^s + A^* X^{-n} A = P$

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In this paper, the Hermitian positive definite solutions of the matrix equation $X^s + A^* X^{-n} A = P$ is considered, where $A$ is an $m \times m$ nonsingular matrix and $P$ is an $m \times m$ Hermite positive definite matrix, $s$ is a positive real number, $n$ is a natural number. Necessary and sufficient conditions for the existence of an Hermitian positive definite solution are derived, and an iterative solution is provided.

Keywords: Matrix Equation, Hermitian positive Definite Solution, iterative Solution

Introduction

We consider the matrix equation

$$X^s + A^* X^{-n} A = P \tag{1.1}$$

where $A$ is an $m \times m$ nonsingular matrix and $P$ is an $m \times m$ Hermite positive definite matrix, $s$ is a positive real number, $n$ is a natural number. We mainly discuss the symmetric positive definite solutions of Eq. (1.1).

Solvability Conditions and Iterative Solution

We consider the following two polynomial equations

$$x^{s+n} - \lambda_{\min}(P)x^n + \lambda_{\max}(A^* A) = 0 \tag{2.1}$$

and

$$x^{s+n} - \lambda_{\max}(P)x^n + \lambda_{\min}(A^* A) = 0 \tag{2.2}$$
Let \( g(x) = \lambda_{\min}^s (P) x^n - x^{s+n} \), \( \xi_* = \frac{1}{\lambda_{\min}^s (P)(\frac{n}{n+s})^\frac{1}{s}} \),
\[
\|A\|^s \|P^{-1}\|^{\frac{n}{n+s}} \leq (\frac{n}{n+s})^\frac{s}{n+s}
\]

It is easy to verify that the necessary and sufficient condition for the existence of the positive real root of Eq. (2.1) and Eq. (2.2) is

\[
\lambda_{\max} (A^* A) < \lambda_{\min} (P) \xi_*^n - \xi_*^{s+n}
\]

\[
= \lambda_{\min} (P) [\lambda_{\min}^s (P)(\frac{n}{n+s})^\frac{1}{s}]^n - \lambda_{\min}^s (P)(\frac{n}{n+s})^\frac{s+n}{s}
\]

\[
= \lambda_{\min}^s (P)[(\frac{n}{n+s})^\frac{n}{s} - (\frac{n}{n+s})^\frac{s+n}{s}]
\]

\[
= \lambda_{\min}^s (P)(\frac{n}{n+s})^\frac{n}{s} \frac{s}{n+s}
\]

\[
= \xi_*^s \left( \frac{1}{n} \frac{s}{n+s} \right)
\]

\[
= \xi_*^s \frac{s}{n}
\]

Thus, in this subsection we assume that \( A \) satisfies

\[
(2.3)
\]

By (2.3) we know that Eq.(2.1) has two positive real roots \( \alpha_2 < \beta_1 \), Eq.(2.2) has two positive real roots \( \alpha_1 < \beta_2 \). It is easy to prove that,

\[
0 < \alpha_1 < \alpha_2 < \xi_* < \beta_1 < \beta_2
\]

**Theorem 2.1:** Suppose that \( A \) and \( P \) satisfies(2.3), \( X \) is the solution of Eq.(1.1). Then

\[
\alpha_1 \leq \lambda_{\min} (X) \leq \alpha_2 \text{ or } \beta_1 \leq \lambda_{\min} (X) \leq \beta_2
\]

\[
\alpha_1 \leq \lambda_{\max} (X) \leq \alpha_2 \text{ or } \beta_1 \leq \lambda_{\max} (X) \leq \beta_2
\]

**Proof:** Suppose \( X \) is the solution of Eq.(1.1), then \( X^s + A^* X^{-n} A = P \), by Wyle inequation we know that,

\[
\lambda_{\min}^s (X) = \lambda_{\min} (P - A^* X^{-n} A) \geq \lambda_{\min} (P) - \lambda_{\max} (A^* X^{-n} A) \geq \lambda_{\min} (P) - \frac{\lambda_{\max} (A^* A)}{\lambda_{\min} (X)}.
\]
\[ \lambda^s_{\max}(X) = \lambda^s_{\max}(P - A^* X^{-n} A) \leq \lambda^s_{\max}(P) - \lambda^s_{\min}(A^* X^{-n} A) \geq \lambda^s_{\max}(P) - \frac{\lambda^s_{\min}(A^* A)}{\lambda^s_{\max}(X)} \]

so, \( \lambda^s_{\min}(X) \geq \beta_1 \) or \( \lambda^s_{\min}(X) \leq \alpha_2 \); \( \alpha_1 \leq \lambda_{\max}(X) \leq \beta_2 \).

On the other hand, \( X^n + A^* X^{-n} A = P \), so \( X^n = A(P - X^*)^{-1} A^* \), then

\[ \frac{A^* A}{\lambda_{\min}(P - X^* \lambda_{\max}(P - X^*)} \geq A(P - X^*)^{-1} A^* = X^n \geq \frac{AA^*}{\lambda_{\max}(P - X^*)} \]

thus \( \lambda^n_{\min}(X) \lambda_{\max}(P - X^*) \geq \lambda^n_{\min}(AA^*) \), \( \lambda^n_{\max}(X) \lambda_{\min}(P - X^*) \leq \lambda^n_{\max}(AA^*) \)

by Wyle inequality, we can obtain

\[ \lambda_{\max}(P - X^*) \leq \lambda_{\max}(P) - \lambda^n_{\min}(X), \lambda_{\min}(P - X^*) \geq \lambda_{\min}(P) - \lambda^n_{\max}(X) \]

so, \( \lambda_{\max}(P) \lambda^n_{\min}(X) - \lambda^n_{\min}(A^* A), \lambda_{\min}(P) \lambda^n_{\max}(X) - \lambda^n_{\max}(A^* A) \)

so, \( \alpha_1 \leq \lambda_{\min}(X) \leq \beta_2 \), \( \lambda_{\max}(X) \geq \beta_1 \) or \( \lambda_{\max}(X) \leq \alpha_2 \).

**Remark 2.1:** Suppose that \( A \) and \( P \) satisfies (2.3), \( X \) is the solution of Eq.(1.1), there are no solutions in \([\alpha_2 I, \beta_1 I] \).

**Remark 2.2:** Suppose that \( A \) and \( P \) satisfies (2.3), \( X \) is the solution of Eq.(1.1), then

\[ X \in [\alpha_1 I, \alpha_2 I] \cup [\beta_1 I, \beta_2 I] \cup \{ X^* = X \mid \alpha_1 \leq \lambda_{\min}(P) \leq \alpha_2, \beta_1 \leq \lambda_{\max}(P) \leq \beta_2 \} \]

**Theorem 2.2:** Suppose that \( A \) and \( P \) satisfies (2.3), We define sequences as the following,

\[ \hat{X}_0 = \gamma I \in [\alpha_2 I, \beta_1 I], \ X_{k+1}^\wedge = [A(P - X_k^\wedge)^{-1} A^*]^\wedge, \ k = 0,1,2... \]

\[ \hat{X}_0 = \gamma I \in [0, \alpha_1 I], \ X_{k+1} = [A(P - X_k)^{-1} A^*]^\wedge, \ k = 0,1,2... \]

the sequences satisfy \( \hat{X}_0 \leq \hat{X}_1 \leq \cdots \leq \hat{X}_n \leq \cdots \leq \hat{X} \leq \cdots \leq \hat{X}_n \leq \cdots \leq \hat{X}_1 \leq \hat{X}_0 \),

then there is a maximal solution \( \hat{X} \) and a minimal solution \( X \),

\[ \hat{X} = \lim_{n \to \infty} \hat{X}_n, \ X = \lim_{n \to \infty} X_n . \]
Proof: Let \( h(X) = [A(P - X^*)^{-1} A^*]^\frac{1}{n} \), \( \Phi = [\alpha_1, \alpha_2 I] \), then \( \forall X \in \Phi \), \( P - X^* > 0 \).

\[
\lambda_{\min}(h(X)) = \lambda_{\min}^n [A(P - X^*)^{-1} A^*] \geq \left[ \frac{\lambda_{\min}(AA^*)}{\lambda_{\max}(P) - \alpha_1^2} \right] \frac{1}{n} = \alpha_1
\]

\[
\lambda_{\max}(h(X)) = \lambda_{\max}^n [A(P - X^*)^{-1} A^*] \leq \left[ \frac{\lambda_{\max}(AA^*)}{\lambda_{\min}(P) - \alpha_2^2} \right] \frac{1}{n} = \alpha_2
\]

So, \( h(\Phi) \subset \Phi \).

\( \forall X, Y \in \Phi, X > Y \), \( h(X) = [A(P - X^*)^{-1} A^*]^\frac{1}{n} > [A(P - X^*)^{-1} A^*]^\frac{1}{n} = h(Y) \), that is to say, \( h(X) \) is monotone increasing.

\[
\hat{h}(X_0) = [A(P - X_0^*)^{-1} A^*]^\frac{1}{n} \leq \left[ \frac{\lambda_{\max}(AA^*)}{\lambda_{\min}(P)} \right] \frac{1}{n} \leq \gamma \leq I = \hat{h}(X_0)
\]

\[
\hat{h}(X_0) = [A(P - X_0^*)^{-1} A^*]^\frac{1}{n} \leq \left[ \frac{\lambda_{\min}(AA^*)}{\lambda_{\max}(P)} \right] \frac{1}{n} \leq \gamma \leq I = \hat{h}(X_0)
\]

It is easy to verify the sequences \( \{\hat{X}_n\}, \{X_n\} \) are convergent,

\[
\hat{X} = \lim_{n \to \infty} \hat{X}_n, \ X = \lim_{n \to \infty} X_n.
\]

Collary. Suppose that A and P satisfies (2.3), \( X \) is the solution of Eq. (1.1). If \( X \in [\alpha_1 I, \alpha_2 I] \), then

\[
\hat{X} \leq X \leq \hat{X}.
\]

References


