Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 4, Number 3 (2012), pp. 271–277

http://www.ripublication.com/gjms.htm

Semigroups in which every Proper Quasi-Ideal is a Group

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Abstract

In this paper, we first introduce the concept of Q-semigroups, then we discuss properties and characterizations of Q-semigroups. Finally, we give structural theorem of Q-semigroups.

AMS subject classification: 20M10.
Keywords: Semigroup, Q-semigroup, quasi-ideal, subsemigroup, subgroup, Rees matrix semigroup.

1. Introduction and Preliminary Results

In the theory semigroups, semigroups with a system of subsemigroups is play an important role. G.Pollák and L.Rédei [1] studied semigroup in which every proper subsemigroup is a group. In [2], Š. Schwarz worked on semigroups in which every proper left ideal is a group (we denote: F-semigroup). The same class of semigroups was studied by R.Hrmová [3], Ǵ.Čupona [4] considered first a semigroup S in which every subset $Sx \neq S$ is a group, later, semigroups in which some left ideal is a group [5]. This result is a special case of the results given by A.H.Clifford [6]. In semigroups, quasi-ideals are a generalization of one-sided ideals. The main purpose of this paper is to study semigroups in which every proper quasi-ideal is a group and give it's structural theorems.

Let Q be a nonempty subset of semigroup S, Q is called a *quasi-ideal* of S if $QS \bigcap SQ \subseteq Q$. Semigroup S is a group if and only if S does not contain proper quasi-ideals [7]. If semigroup S is not a group, then S contains proper quasi-ideals. Therefore, we can give following definition.

Definition 1.1. A semigroup S is called *Q-semigroup* if S is not a group but every proper quasi-ideal of S is a group.

Lemma 1.2. [7] The minimal quasi-ideals of semigroup S are just the maximal subgroups of the kernel of S.

Lemma 1.3. Let Q be a quasi-ideal of semigroup S and G a subgroup of semigroup S. Then the following statements hold.

(1) $G \bigcap Q \neq \Omega$, then $G \subseteq Q$.

(2) If Q is a minimal quasi-ideal of semigroup S, then G contains in the kernel of S.

Proof. (1) Let $a \in G \bigcap Q$. Since G is a group, we have $G = Ga \bigcap aG$. From this it follows that $G = Ga \bigcap aG \subseteq Sa \bigcap aS \subseteq SQ \bigcap QS \subseteq Q$.

(2) By Lemma 1.2, it is obvious. ■

Lemma 1.4. Let Q_1, Q_2 be two quasi-ideals of Q-semigroup S, then one of the following statements hold.

(1) $Q_1 \bigcap Q_2 = \Omega$.

(2) $Q_1 = Q_2$.

Proof. Since S is a Q-semigroup, then Q_1, Q_2 are subgroups of S. If $Q_1 \bigcap Q_2 \neq \Omega$, by Lemma 1.2, then $Q_1 \subseteq Q_2$ and $Q_2 \subseteq Q_1$. Hence, $Q_1 = Q_2$. ■

Remark 1.5. In Q-semigroup S, if Q is a proper quasi-ideal of S, then Q is a minimal quasi-ideal of S. In fact: Let Q_1 is a nonempty quasi-ideal of S such that $Q_1 \subseteq Q$, then $Q_1 \bigcap Q \neq \Omega$. By Lemma 1.4, we have $Q_1 = Q$.

Lemma 1.6. [2] Every proper quasi-ideal of semigroup S is a bi-ideal of S.

Lemma 1.7. [2] A F-semigroup S contains at most two disjoint proper left ideals L_1, L_2 or proper right ideals R_1, R_2 such that $S = L_1 \bigcup L_2 = R_1 \bigcup R_2$.

Lemma 1.8. [8] The following conditions are equivalent on a semigroup S

(1) S has a bi-ideal which is a group.

(2) S contains the kernel K which is a union of groups.

Theorem 1.9. A Q-semigroup S contains at most two disjoint proper quasi-ideals Q_1, Q_2 such that $S = Q_1 \bigcup Q_2$.

Proof. Let Q-semigroup S contains more than two different proper quasi-ideals $Q_a, Q_b, Q_c, \cdots$, then $Q_a, Q_b, Q_c, \cdots$ are subgroups of S, by Lemma 1.4, $Q_a, Q_b, Q_c, \cdots$ are disjoint. Now let

$$M = Q_a \bigcup Q_b \bigcup Q_c \bigcup \cdots$$

by Lemma 1.6, $Q_a, Q_b, Q_c, \cdots$ are bi-ideals of S and they are subgroups of S. In view of Lemma 1.8, S has a kernel K which is a union of subgroups of S. From this it follows that $M \subseteq K$. Conversely, since K is a ideal of S, therefore, K is a quasi-ideal of S, we have $K \subseteq M$. Thus $M = K$. We obtain M is a quasi-ideal of S.

If $M \neq S$, then M is a proper quasi-ideal of S, so M is a subgroup of S. By Lemma 1.6, we conclude that M is a bi-ideal of S and it is a minimal bi-ideal of S, but $Q_i \subset M (i = a, b, c, \cdots)$, which is contradiction. Hence, we have $M = S = Q_a \bigcup Q_b \bigcup Q_c \bigcup \cdots$. Since a Q-semigroup is a F-semigroup, from this it follows that S contains at most two disjoint proper left ideals or disjoint proper right ideals. Without loss of generality, we can assume that $S = Q_1 \bigcup Q_2$, Q_1, Q_2 are two disjoint proper left ideals of S. Then we have

$$Q_1 \bigcup Q_2 = Q_a \bigcup Q_b \bigcup Q_c \bigcup \cdots.$$

Since Q_1, Q_2 are two proper left ideals of S, then Q_1, Q_2 are two proper quasi-ideals of S, so that Q_1, Q_2 are two subgroups of S. But $Q_i \subset M (i = a, b, c, \cdots)$ is disjoint and it is a minimal quasi-ideal of S. Hence Q_1, Q_2 certainly coincide with two quasi-ideals of $Q_a, Q_b, Q_c, \cdots$. Without loss of generality, we can assume that $Q_1 = Q_a$, $Q_2 = Q_b$, then $Q_c = Q_d = \cdots = \Omega$, which shows that S contains at most two different proper quasi-ideals. ■

Theorem 1.10. A semigroup S is a Q-semigroup if and only if one of the following statements hold.

(1) S contains unique a proper quasi-ideal.

(2) S contains exactly two proper quasi-ideals Q_1, Q_2 such that $S = Q_1 \bigcup Q_2$ and $Q_1 \bigcap Q_2 = \Omega$.

Proof. $\Leftarrow$) Let S contains exactly one proper quasi-ideal Q, then Q is a minimal quasi-ideal of S. By Lemma 1.1, Q is a maximal subgroups of the kernel of S. But S is not a group and S has proper quasi-ideal, which shows that S is a Q-semigroup.

If condition (2) hold, for two proper quasi-ideals Q_1, Q_2 of S, since $Q_1 \bigcap Q_2 = \Omega$, so that Q_1, Q_2 are minimal quasi-ideals of S. By Lemma 1.1, Q_1, Q_2 are two maximal subgroups of the kernel of S. Since S has proper quasi-ideals and S is not a group, thus S is a Q-semigroup.

$\Rightarrow$) It follows immediately from Lemma 1.3 and Theorem 1.9. ■

2. Structural Theorems

Let G be a group and I and Λ nonempty sets in which elements are denoted by $i, j, k, \cdots$ and $\lambda, \mu, \nu, \cdots$ respectively. Let P be a matrix of the type $\Lambda \times I$ with elements $p_{\lambda i}$ from G. On a set $S = G \times I \times \Lambda$, we define a multiplication with

$$(a; i, \lambda)(b; j, \mu) = (ap_{\lambda j}b; i, \mu)$$

Then S with this multiplication is a semigroup which we shall call the *Rees matrix semigroup* over the group G with a sandwich matrix P and denoted by $M(G; I, \Lambda, P)$.

Now we consider L-class set and R-class set of $M(G; I, \Lambda, P)$:

$$\{L_\lambda | \lambda \in \Lambda\}, \quad L_\lambda = \{(a; i, \lambda) | a \in G, i \in I\}$$

$$\{R_i | i \in I\}, \quad R_i = \{(a; i, \lambda) | a \in G, \lambda \in \Lambda\}$$

For an easy way check that L_λ is a left ideal of $M(G; I, \Lambda, P)$ and R_i a right ideal of $M(G; I, \Lambda, P)$, so that L_λ and R_i are quasi-ideals of $M(G; I, \Lambda, P)$ and $L_{\lambda_1} \bigcap L_{\lambda_2} = \Omega, R_{i_1} \bigcap R_{i_2} = \Omega, \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \neq \lambda_2, i_1, i_2 \in I, i_1 \neq i_2$.

Theorem 2.1. Let $M(G; I, \Lambda, P)$ be a Rees matrix semigroup over the group G. If $|I| = 1, \Lambda = \{\lambda_1, \lambda_2\}$, then $L_{\lambda_1}, L_{\lambda_2}$ are subgroups of $M(G; I, \Lambda, P)$.

Proof. For L_{λ_1}, it is obvious that $e = (p_{\lambda_1 i}^{-1}; i, \lambda_1)$ is exactly one idempotent of L_{λ_1}. Let $(a; i, \lambda_1) \in L_{\lambda_1}$, then we have

$$(a; i, \lambda_1)(p_{\lambda_1 i}^{-1}; i, \lambda_1) = (p_{\lambda_1 i}^{-1}; i, \lambda_1)(a; i, \lambda_1) = (a; i, \lambda_1).$$

For $(a; i, \lambda_1)$, there exists $(p_{\lambda_1 i}^{-1} a^{-1} p_{\lambda_1 i}^{-1}; i, \lambda_1) \in L_{\lambda_1}$ such that

$$(a; i, \lambda_1)(p_{\lambda_1 i}^{-1} a^{-1} p_{\lambda_1 i}^{-1}; i, \lambda_1) = (p_{\lambda_1 i}^{-1}; i, \lambda_1) = (p_{\lambda_1 i}^{-1} a^{-1} p_{\lambda_1 i}^{-1}; i, \lambda_1)(a; i, \lambda_1)$$

which shows that L_{λ_1} is a group. In a similar way we can prove that L_{λ_2} is a group, too.

Symmetrically, If $|I| = \{i_1, i_2\}, |\Lambda| = 1$, then R_{i_1}, R_{i_2} are also subgroups of $M(G; I, \Lambda, P)$. ■

Theorem 2.2. Let $S = M(G; I, \Lambda, P)$. If $|I| = 1, |\Lambda| = 2$ or $|I| = 2, |\Lambda| = 1$, then S is a Q-semigroup and $S = \bigcup_{\lambda \in \Lambda} L_\lambda$ or $S = \bigcup_{i \in I} R_i$.

Proof. Let $|I| = 1, \Lambda = \{\lambda_1, \lambda_2\}$, then $L_{\lambda_1} \bigcap L_{\lambda_2} = \Omega$ and $L_{\lambda_1}, L_{\lambda_2}$ are subgroups of S and proper quasi-ideals of S. Let Q is a proper quasi-ideal of S. For each $(a; i, \lambda) \in Q$. If $\lambda = \lambda_1$, then $(a; i, \lambda) \in L_{\lambda_1}$. If $\lambda = \lambda_2$, then $(a; i, \lambda) \in L_{\lambda_2}$. Hence we have $Q \subseteq L_{\lambda_1}$ or

$Q \subseteq L_{\lambda_2}$, from this it follows that $Q \bigcap L_{\lambda_1} \neq \Omega$ or $Q \bigcap L_{\lambda_2} \neq \Omega$. If $Q \bigcap L_{\lambda_1} \neq \Omega$, then for each $(a; i, \lambda) \in Q \bigcap L_{\lambda_1}$, since L_{λ_1} is group of S, we have that

$$L_{\lambda_1} = (a; i, \lambda)L_{\lambda_1} \bigcap L_{\lambda_1}(a; i, \lambda) \subseteq (a; i, \lambda)S \bigcap S(a; i, \lambda) \subseteq QS \bigcap SQ \subseteq Q$$

which shows that $Q = L_{\lambda_1}$. If $Q \bigcap L_{\lambda_2} \neq \Omega$, in the same way, we have $Q = L_{\lambda_2}$. Hence S contains exactly two proper quasi-ideals $L_{\lambda_1}, L_{\lambda_2}$ and $L_{\lambda_1}, L_{\lambda_2}$ is subgroups of S, so that S is a Q-semigroup. It is easy to prove that $S = \bigcup\limits_{\lambda \in \Lambda} L_\lambda$.

For $|I| = 2, |\Lambda| = 1$, in a similar way we can prove that $S = \bigcup\limits_{i \in I} R_i$.

If S is a nonempty set and " $\cdot$ " is an operation defined for some elements of S, then " $\cdot$ " is a partial operation. If for $x, y, z \in S$ there exist $x \cdot (y \cdot z)$, $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, then S is a *partial semigroup*. ∎

Theorem 2.3. Let G be a group, P a partial semigroup, $G \bigcap P = \Omega$ and $\phi : P \longrightarrow G$ a partial homomorphism. Let us extend ϕ to a mapping $\psi : S = G \bigcup P \longrightarrow G$ by $\psi(x) = \phi(x)$ if $x \in P$ and $\psi(a) = a$ for all $a \in G$. Let us define an operation on S by

$$xy = \begin{cases} \phi(x)\phi(y), & \text{if } x, y \in P \text{ and } xy \text{ is defined in } P \\ \psi(x)\psi(y), & \text{otherwise} \end{cases}$$

then S is a Q-semigroup with the unique proper quasi-ideal G. We shall denote the semigroup S constructed above by $S = M(G, P, \phi, \psi)$.

Proof. For an easy way we can prove that S is a semigroup. By definition of operation on S, we have $SG \subseteq G, GS \subseteq G$, so $GS \bigcap SG \subseteq G$. In view of $G \neq S$, then G is a proper quasi-ideal of S. Let Q is a proper quasi-ideal of S. Since $G \bigcap P = \Omega$, we have $Q \bigcap G \neq \Omega$ or $Q \bigcap P \neq \Omega$. If $Q \bigcap G \neq \Omega$, since G is a subgroup of S and quasi-ideal of S, by Lemma 1.2, we have $G \subseteq Q$, so that $S = G \bigcup P \subseteq Q \bigcup P$. Hence $S = G \bigcup P = Q \bigcup P$, and from this it follows that $G = Q$. If $Q \bigcap P \neq \Omega$, then $P \subseteq Q$ or $Q \subseteq P$. In fact: If $Q \bigcap G \neq \Omega$, the proof is the same as in above, we have $G = Q$, this is a contradiction. If $P \subseteq Q$, then $S = G \bigcup P \subseteq G \bigcup Q$, so $G \bigcup P = G \bigcup Q = S$, and from this it follows that $P = Q$. But P is a partial semigroup and not quasi-ideal of S, hence $P \neq Q$. Therefore we have $Q \subset P$. Since Q is a quasi-ideal of S, let $x \in Q \bigcap P$, then there exists $a, b \in S$ such that $ax = xb \in SQ \bigcap QS \subseteq Q \subset P$. Since $ax = xb \in G$, we have $G \bigcap P \neq \Omega$, this is a contradiction. Hence $Q \bigcap P = \Omega$. Moreover, Since S is not a group, so S is a Q-semigroup with the unique proper quasi-ideal G. ∎

Theorem 2.4. A semigroup S is a Q-semigroup if and only if one of the following statements hold.

(1) S is isomorphic to some $M(G; I, \Lambda, P), |I| = 1, |\Lambda| = 2$ or $|I| = 2, |\Lambda| = 1$.

(2) S is isomorphic to some $M(G, P, \phi, \psi)$.

Proof. Let S be a Q-semigroup. By Theorem 1.1, S contains unique a proper quasi-ideals or S contains exactly two disjoint proper quasi-ideals.

Case 1: Let S contains exactly two disjoint proper quasi-ideals Q_1, Q_2 and $S = Q_1 \bigcup Q_2$, and Q_1, Q_2 are groups. Without loss of generality, we can assume that $|I| = 1, |\Lambda| = 2$. We consider the mapping

$$\phi_1 : Q_1 \longrightarrow L_{\lambda_1} | \phi_1(a) = (aP^{-1}_{\lambda_1 i}; i, \lambda_1)$$

$$\phi_2 : Q_2 \longrightarrow L_{\lambda_2} | \phi_2(b) = (bP^{-1}_{\lambda_2 i}; i, \lambda_2)$$

it is not hard to check that the mapping ϕ_1, ϕ_2 are isomorphic. Let us defined a mapping ω on S by

$$\omega : S \longrightarrow M(G; I, \Lambda, P) | \omega|_{Q_1} := \phi_1, \omega|_{Q_2} := \phi_2, (ab)\omega = a\phi_1 \cdot b\phi_2, a \in Q_1, b \in Q_2$$

it is easily prove that ω is isomorphic.

Case 2: Let S contains unique a proper quasi-ideals Q, then Q is a subgroup of S. Let $G = Q, P = S - G$, then P is a partial semigroup, from this it follows that S belong to the class of semigroup $M(G, P, \phi, \psi)$.

Conversely, it follows from Theorem 2.2 and Theorem 2.3. ■

References

[1] G. Pollák, L. Redei, Die halbgruppen, deren alle echten teilhabgruppen gruppen sind, Publ. Math. Debrecen, 6(1959), 125–131.

[2] Š. Schwarz, Semigroups in which every proper subideal is a group, Acta Sci. math. Szeged, 21(1960), 125–131.

[3] R. Hrmová, Polugrupý v ktorých kaźdý lývývlastný ideál je groupu, Math. Fiz. C̄asop., 11(1961), 75–80.

[4] Ǵ.Čupona, On semigroups S in which each proper subset Sx is a group, Glasnik Math. Fiz. Astr., 18(1963).

[5] Ǵ. Čupona, On semigroups S in which some left ideal is a group, God. Zbornik PMF, Kopje, T., 14(1963), 15–17.

[6] A.H. Clifford, Extensions of semigroups, Trans. Amer. Math. Soc., 68(1950), 165–173.

[7] A.H.Clifford, Preston G.B., The algebraic theory of semigroups I, Math Surveys, Math. Soc. Providence, 1961.

[8] Stojan Bogdanović, Semigroups with a system of subsemigroups, University of Novi Sad Institute of Math. Novi Sad, 1985.