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Dynamics of Linear Bidirectional Synchronization of Shimizu-Morioka chaotic system

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Abstract

The Shimizu-Morioka dynamical system is a Lorentz-like system that is of much importance in fields like fluid dynamics and laser physics. This paper proposes an identical synchronization scheme for generalized linearly bidirectionally coupled chaotic Shimizu-Morioka dynamical system. Lyapunov stability theory is applied to establish the conditions on coupling parameters for synchronization. Numerical simulation results are presented to establish the feasibility and effectiveness of the approach.

AMS subject classification:
Keywords: Shimizu-Morioka dynamical system, Chaos Synchronization, Lyapunov Stability theory

1. Introduction

The concept of chaos synchronization was first introduced in 1990 by Pecora *et al.*[1]. Since then chaos control and synchronization has become an important subject in the field of non-linear science due to its potential applications in different disciplines. Various synchronization and control methods such as, impulsive control method [2], active control method [3], adaptive control method [4], linear and non-linear feedback control method [5], unidirectional and bidirectional synchronization of coupled systems [6, 7, 8] have been successfully applied to chaos synchronization.

A pair of coupled dynamical systems is known as unidirectional if one system is independent but the second system depends on the first. If $\vec{X}$ and $\vec{Y}$ denote the state vectors of the two systems then the mathematical forms of the unidirectionally coupled systems are $\dot{\vec{X}} = f(\vec{X})$ and $\dot{\vec{Y}} = g(\vec{X}, \vec{Y})$. Physically it means that in some parts of the phase space, the behaviour of one system is influenced by the behavior of the other, but the driving system is totally independent of the responses of the responding system. If the coupling is bidirectional, both the systems are interdependent and phase space behaviour of one of the system directly influences the behaviour of the other and vice versa. Mathematical forms of the bidirectionally coupled systems are $\dot{\vec{X}} = f_1(\vec{X}, \vec{Y})$ and $\dot{\vec{Y}} = g_1(\vec{X}, \vec{Y})$.

Most of the natural systems are bidirectionally coupled and therefore the study of bidirectionally coupled systems is necessary. In this paper we have studied the linear bidirectional coupling on the Shimizu- Morioka chaotic dynamical system [6] and derived the restrictions on the coupling parameters for the synchronization using Lyapunov stability theory.

2. The Shimizu-Morioka Chaotic System

Shimizu-Morioka dynamical system[9] is described by the system of differential equations:

$$
\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= x_1 - \lambda x_2 - x_1 x_3 \\
\dot{x_3} &= \alpha x_3 + x_1^2
\end{aligned}
\tag{1}
$$

This model received much attention due to its ability to describe bifurcation of the associated Lorentz-like attractors. Studies by Shil'nikov[10] revealed the boundary region of existence of Lorentz-like attractors to be two points $Q_a(\alpha = 0.608, \lambda = 1.0499)$ and $Q_A(\alpha = 0.549, \lambda = 0.605)$ in the two-dimensional parametric space.
This system can be written as

$$
\dot{\vec{X}} = A\vec{X} + \phi(\vec{X}) \tag{2}
$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ and $\phi(\vec{X}) = \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1^2 \end{pmatrix}$.

Let us consider two Shimizu-Morioka hyper-chaotic bidirectionally coupled systems:

$$\dot{\vec{X}} = A\vec{X} + \phi(\vec{X}) + B(\vec{Y} - \vec{X}) \tag{3}$$

$$\dot{\vec{Y}} = A\vec{Y} + \phi(\vec{Y}) + B(\vec{X} - \vec{Y}) \tag{4}$$

where $Y = (y_1, y_2, y_3)^T$ and

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix};$$

b_{ij}' s being coupling coefficients, the matrix B is known as the generalised bidirectional coupling coefficient matrix. Coupling is called linear if b_{ij}' s are constants and non-linear if they are functions of the state variables. Systems (3) and (4) synchronize asymptotically if

$$\lim_{t\to\infty} \parallel Y(t) - X(t) \parallel = 0 \tag{5}$$

where $\parallel . \parallel$ denotes the Euclidean norm. In this paper, we find suitable b_{ij}' s such that (5) is satisfied.

3. Synchronization of linearly bidirectionally coupled Shimizu-Morioka systems

Let us define the synchronization error $\bar{e}(t) = \vec{Y}(t) - \vec{X}(t)$. Using equations (3) and (4) one arrives at the dynamical system of the synchronization error as

$$\dot{\bar{e}} = A\bar{e} + \phi(\vec{Y}) - \phi(\vec{X}) - 2B\bar{e} \tag{6}$$

It can be written as

$$\dot{\bar{e}} = (A + D - 2B)\bar{e} \tag{7}$$

where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ -y_3 & 0 & -x_1 \\ y_1 + x_1 & 0 & 0 \end{pmatrix} \tag{8}$$

Let us define a Lyapunov function $v(\bar{e}) = \frac{1}{2}\bar{e}^T\bar{e}$, which is clearly positive definite.

It yields $\frac{d}{dt}\{v(\bar{e})\} = \bar{e}^T(Q_1 + Q_2)\bar{e}$, where

$$Q_1 = \begin{pmatrix} -2b_{11} & 1-(b_{12}+b_{21}) & -(b_{13}+b_{31}) \\ 1-(b_{12}+b_{21}) & -(\lambda+2b_{22}) & -(b_{23}+b_{32}) \\ -(b_{13}+b_{31}) & -(b_{23}+b_{32}) & \alpha-2b_{33} \end{pmatrix}$$
$$and\ Q_2 = \begin{pmatrix} 0 & y_3/2 & \{y_1+x_1\}/2 \\ -y_3/2 & 0 & -x_1/2 \\ \{y_1+x_1\}/2 & -x_1/2 & 0 \end{pmatrix} \tag{9}$$

The characteristic equation of Q_2 is

$$f(k) \equiv k^3 - \frac{1}{4}k\{x_1^2 + y_3^2 + (x_1+y_1)^2\} + \frac{1}{4}x_1y_3(x_1+y_1) = 0 \tag{10}$$

Simple mathematical calculations yield that if $\{x_1^2+y_3^2+(x_1+y_1)^2\}^3 > 27x_1^2y_3^2(x_1+y_1)^2$, all the roots of (10) will be real. If all the roots of $f(k)=0$ are real, they will be separated by roots of $f'(k)=0$. hence one root of $f(k)=0$ is greater than β and one root is less than $-\beta$, where $\beta = \frac{1}{2\sqrt{3}}\{x_1^2+y_3^2+(x_1+y_1)^2\}^{1/2} > 0$. The greatest and least roots of $f(k)=0$ lie in the intervals (β,∞) and $(-\infty,-\beta)$ respectively. We now choose $M = max\{|M_1|,|M_2|\}$ such that $\beta < M_1 < \infty$, $-\infty < M_2 < -\beta$ and $f(M_1) > 0$, $f(M_2) < 0$. Hence maximum absolute value of the eigenvalues of the matrix $Q_2 < M$. Phase space of the chaotic system is always bounded and consequently we can always find such a bound M.

Thus, $\frac{d}{dt}\{v(\bar{e})\} = \bar{e}^T(Q_1+Q_2)\bar{e} \le \bar{e}^T(Q_1+MI)\bar{e} = -\bar{e}^TQ\bar{e}$, where

$$Q = \begin{pmatrix} 2b_{11}-M & -1+b_{12}+b_{21} & b_{13}+b_{31} \\ -1+b_{12}+b_{21} & \lambda+2b_{22}-M & b_{23}+b_{32} \\ b_{13}+b_{31} & b_{23}+b_{32} & -\alpha+2b_{33}-M \end{pmatrix}.$$

$\frac{d}{dt}\{v(\bar{e})\}$ is negative definite if Q is positive definite. In this case $v(\bar{e})$ becomes a Lyapunov function for the error dynamical system (7). The symmetric matrix Q is positive definite if all the principal minors are positive. For synchronization, b_{ij}'s are so chosen that:

(i) $2b_{11} - M > 0$

(ii) $(2b_{11}-M)(\lambda+2b_{22}-M) - (b_{12}+b_{21}-1)^2 > 0$

(iii) $det\ Q > 0$

If these conditions are satisfied, then by Lyapunov stability theory $\lim_{t\to\infty} \| \bar{e}(t) \| = 0$.

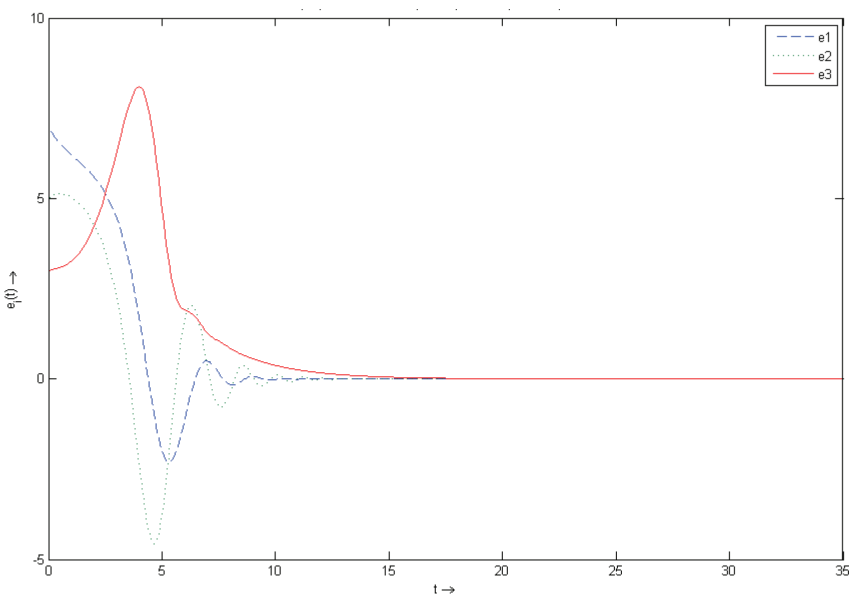

Figure 1: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.45, b_{22} = 0.3$ and $b_{33} = 0.4$

4. Results and Discussion

In this paper we have discussed the identical synchronization of two bidirectionally and linearly coupled Shimizu-Morioka chaotic dynamical systems. To achieve global stability of the synchronized state, we have deduced the restrictions on the coupling coefficients. Applying Lyapunov stability theory, asymptotic stability analysis is made. The deduced restrictions on the coupling parameters are not necessary but sufficient. Numerical simulations are done by fourth order Runge-Kutta method. We have plotted $e_i(t)$ versus t for $i = 1, 2, 3$ simultaneously in each figure to compare the rate of synchronization for different values of the system parameters. It is seen that the synchronisation time depends on the values of the coupling parameters. The Shimizu-Morioka system is a Lorentz-like system that is of much importance in fields like fluid dynamics and laser physics. Thus the results obtained here are expected to be of substantial value for several disciplines.

In all the simulations, we have taken the initial values of the errors as $e_1(0) = 7$, $e_2(0) = 5$, $e_3(0) = 3$ and the values of the system parameters as $\alpha = 0.4$, $\lambda = 0.6$. Also we have considered coupling coefficients $b_{ij} = 0$ for $i \neq j$ and $i, j = 1, 2, 3$.

In fig. 1 and fig. 2 , coupling coefficients $b_{22} = 0.3$ and $b_{33} = 0.4$ are kept fixed but b_{11} differs - which are 0.11 and 0.45 respectively. It is seen that synchronizations occur early for higher values of b_{11}. The same thing is seen in fig. 3 and fig. 4 where we have considered b_{11} equal to 0.3 and 0.45 respectively keeping $b_{22} = 0.45$ and $b_{33} = 0.7$ fixed in each case.

In fig. 5 and fig. 6, we have taken $b_{11} = 0.4$ and $b_{33} = 0.8$ but b_{22} varies. It is 1 in fig. 5 and 0.4 in fig. 6. Synchronization delays as b_{22} decreases. Same pattern is seen in fig. 7,fig. 8 and fig. 9 where we have considered b_{22} equal to 0.8, 0.5 and 2 respectively.

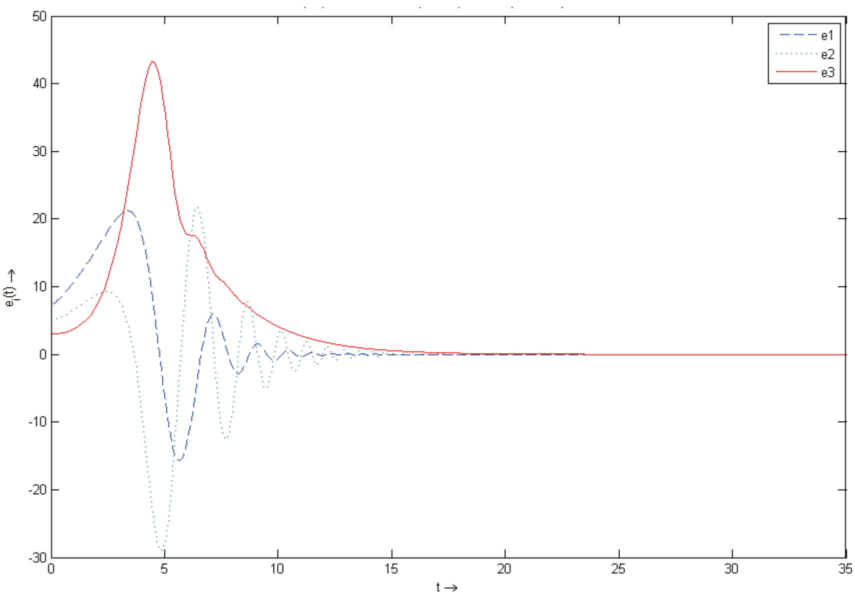

Figure 2: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.11, b_{22} = 0.3$ and $b_{33} = 0.4$

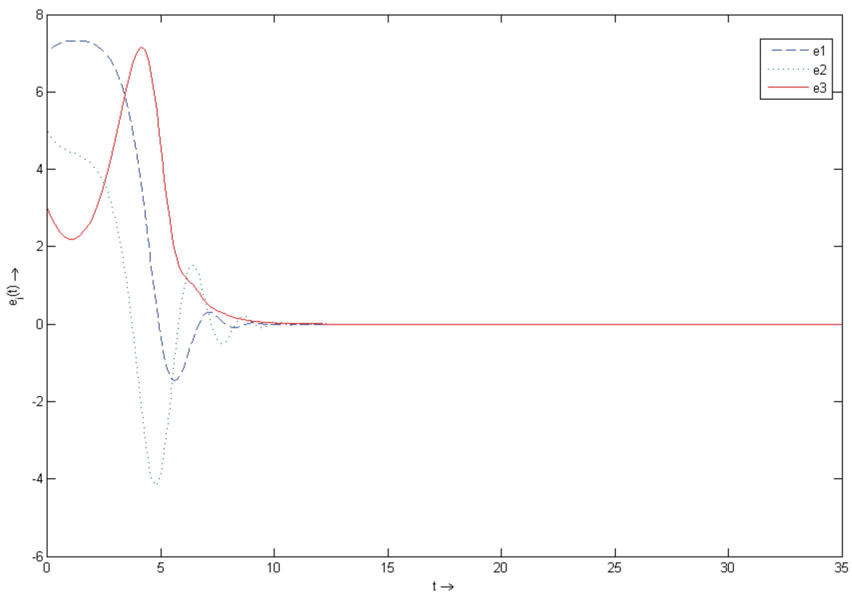

Figure 3: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.3$, $b_{22} = 0.5$ and $b_{33} = 0.7$

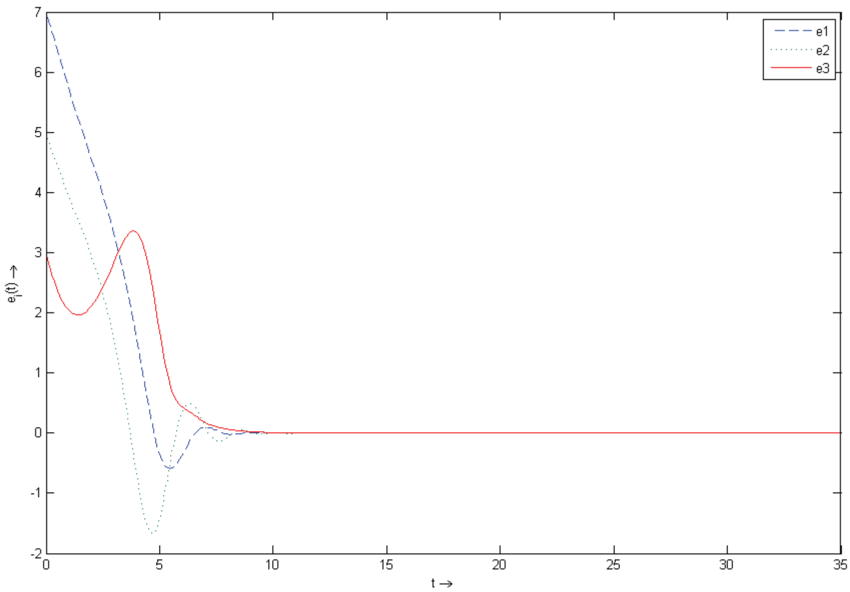

Figure 4: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.45$, $b_{22} = 0.5$ and $b_{33} = 0.7$

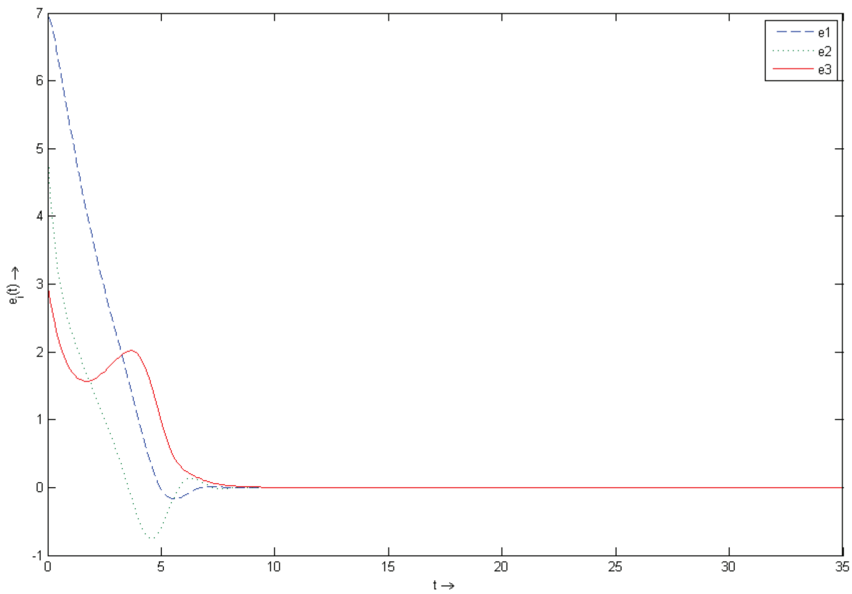

Figure 5: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.4$, $b_{22} = 1$ and $b_{33} = 0.8$

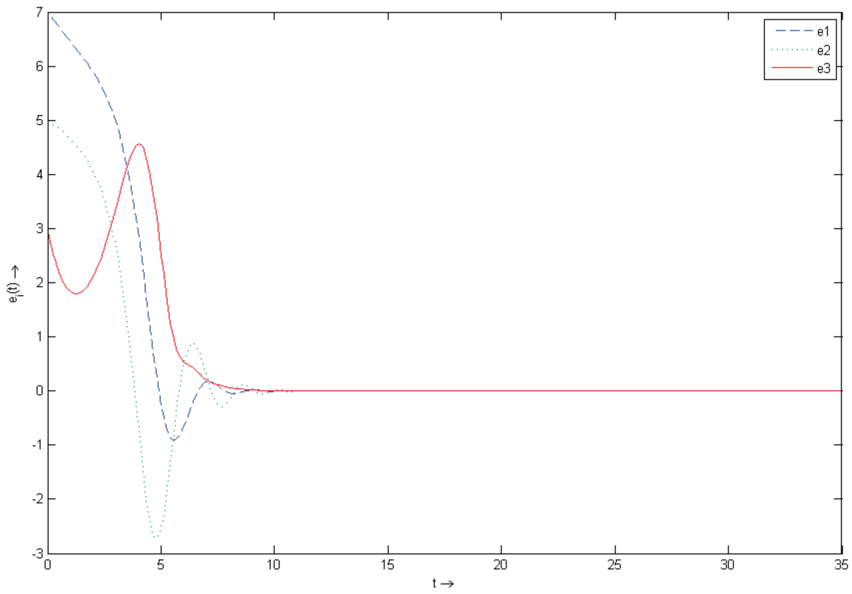

Figure 6: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.4$, $b_{22} = 0.4$ and $b_{33} = 0.8$

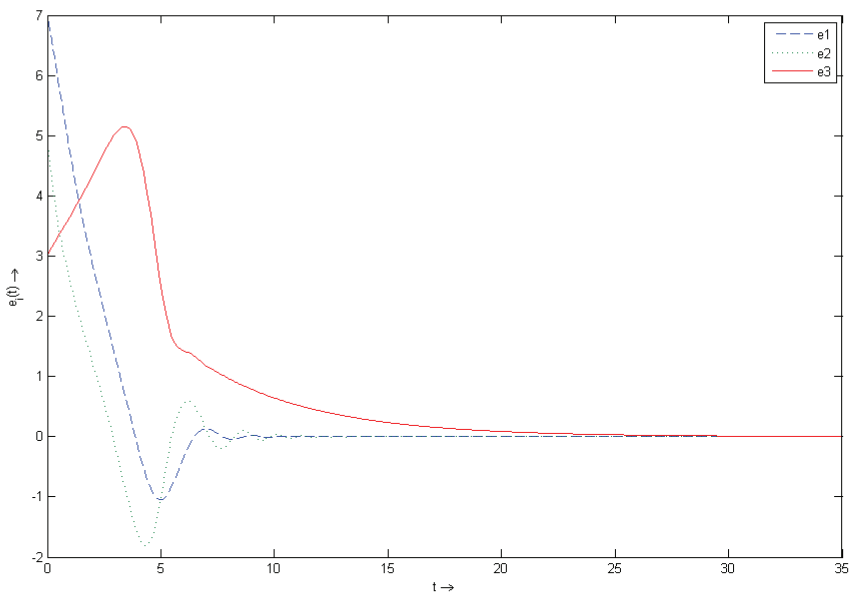

Figure 7: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.5$, $b_{22} = 0.8$ and $b_{33} = 0.3$

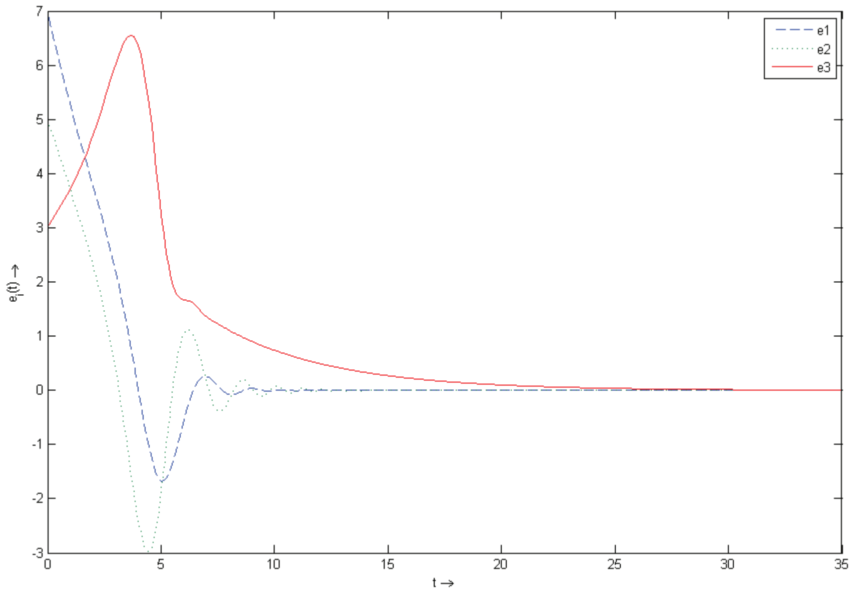

Figure 8: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.5$, $b_{22} = 0.5$ and $b_{33} = 0.3$

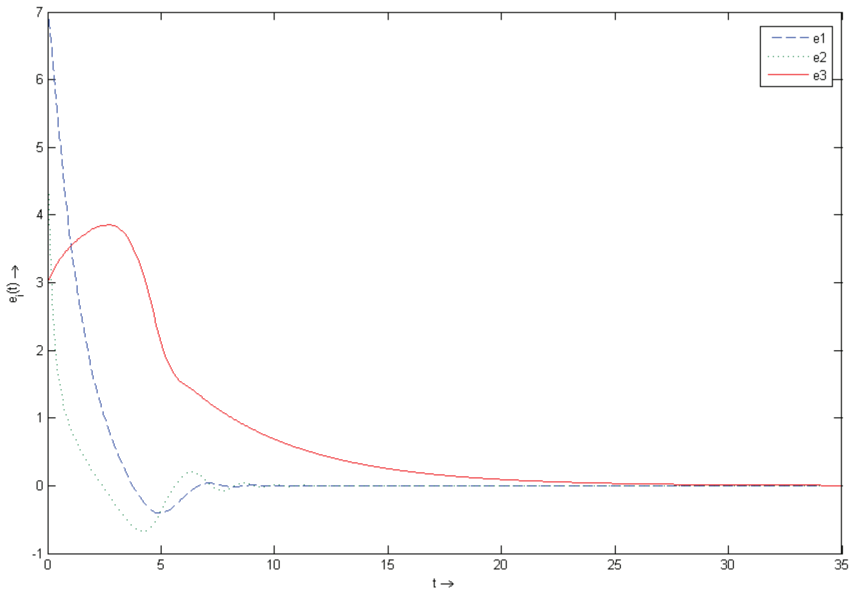

Figure 9: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.5$, $b_{22} = 2$ and $b_{33} = 0.3$

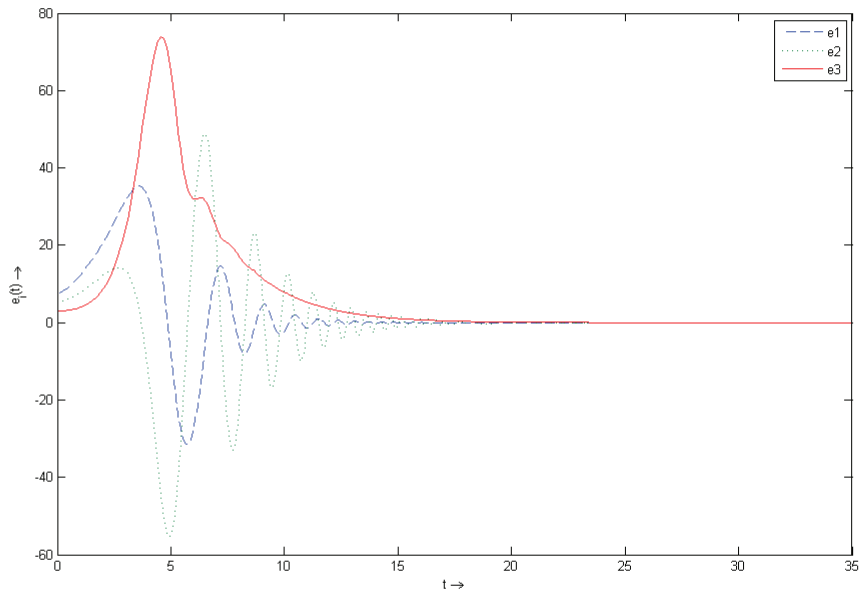

Figure 10: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.05$, $b_{22} = 0.2$ and $b_{33} = 0.4$

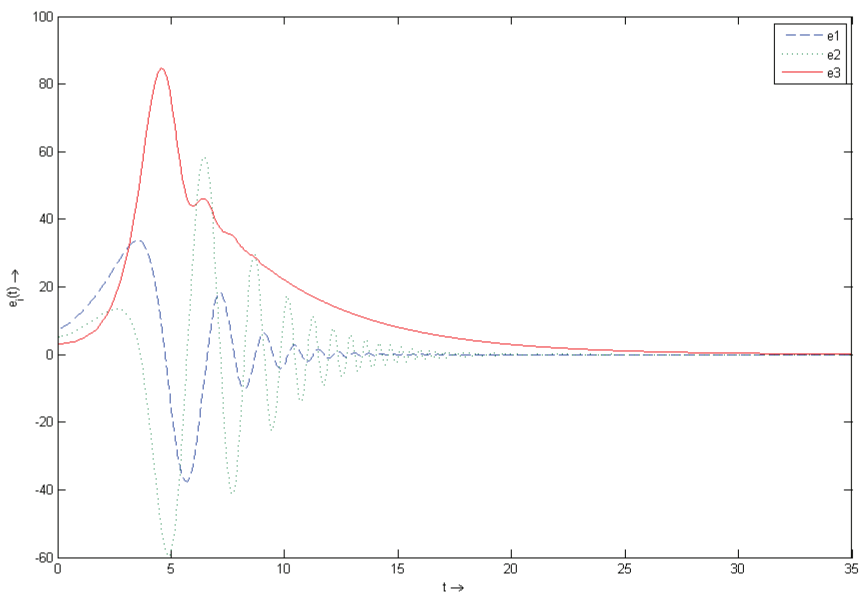

Figure 11: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.05$, $b_{22} = 0.2$ and $b_{33} = 0.3$

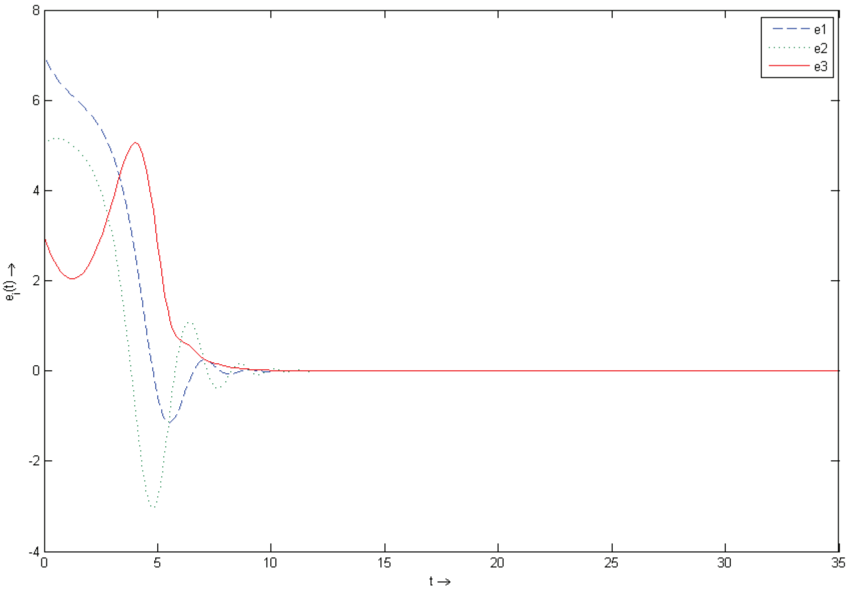

Figure 12: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.45$, $b_{22} = 0.3$ and $b_{33} = 0.7$

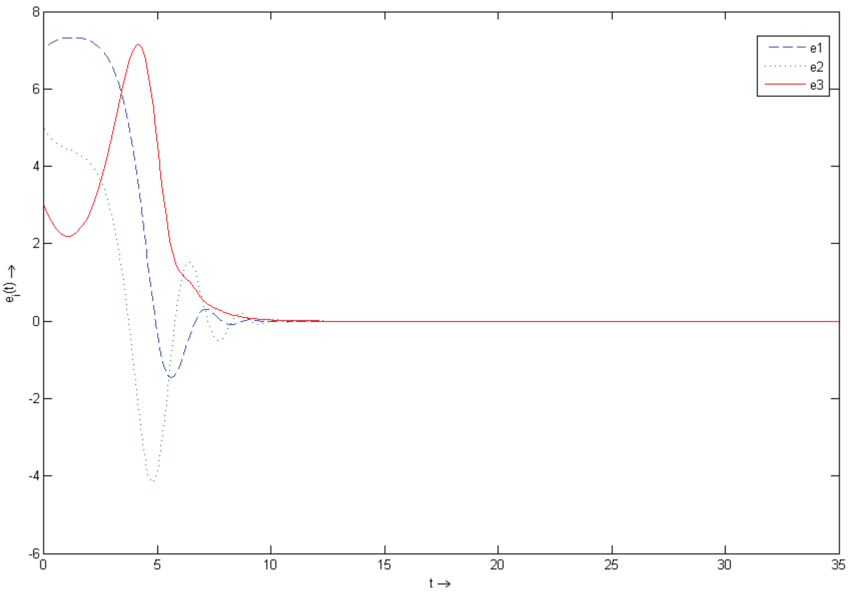

Figure 13: Time evolution of $e_1(t)$, $e_2(t)$ and $e_3(t)$ for $b_{11} = 0.45$, $b_{22} = 0.3$ and $b_{33} = 0.35$

But in each case, $b_{11} = 0.5$ and $b_{33} = 0.3$.

Taking $b_{11} = 0.05$ and $b_{22} = 0.2$, we have plotted fig. 10 and fig. 11 for b_{33} equal to 0.4 and 0.3 respectively and seen that synchronization delays due to the lowering of the values of b_{33}. Same situation occurs when we take $b_{11} = 0.45$ and $b_{22} = 0.3$ and plot fig. 12 and fig. 13 by considering b_{33} equal to 0.7 and 0.35 respectively.

It is also seen from the figures that if $b_{11} < b_{33}$, then high value of b_{22} makes the convergence rate fast but if $b_{11} > b_{33}$, then the rate of convergence does not depend much on the values of b_{22}.

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