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Existence Theory of Second Order Random Differential Equations

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Abstract

In this paper, an existence result for nonlinear second order ordinary random differential equations is proved under a Carath'eodory condition. My investigations is placed in the Banach space of continuous real valued functions on closed and bounded intervals of the real line together with an application of the random version of the Leray-Schauder principle.

Keywords: Nonlinear random differential equation, multi-valued function, local attractivity, global attractivity, boundary value problem.

Mathematics Subject Classifications: 60H25, 47H40, 47N20 .

Statement of the Problem

Let R denote the real line and let $J=[0,T]$ be a closed and bounded interval in R. let $C^1(J,R)$ denote the class of realvalued functions defined and continuously differentiable on J. Given a measurable space (Ω, A) and for a given measurable function $x:\Omega \to C^1(J,R)$, consider the boundary value problem of second order ordinary random differential equations (RDE),

$$\begin{aligned} -x''(t,\omega) &= f(t,x(t,\omega),\omega) \ \ a.e.\, t\in J, \\ x(0,\omega) &= q_0(\omega), x'(0,\omega) = q_1(\omega), \end{aligned} \tag{1.1}$$

for all $\omega \in \Omega$, where $f: J\times R\times \Omega \to R$, $q_0, q_1 : \Omega \to R$.

By a random solution of the RDE. (1.1), I mean a measurable function $x:\Omega \to AC^1(J,R)$ that satisfies the equations in (1.1) where $AC^1(J,R)$ is the space of real valued functions defined and absolutely continuously differentiable on J.

The RDE (1.1) is not new to the theory random differential equations. When the random parameter ω is absent, the RDE (1.1) reduces to the classical RDE of second order ordinary differential equations (ODE),

$$\begin{aligned} -x''(t) &= f(t, x(t)) \ \ a.e. t \in J, \\ x(0) &= x_0, x'(0) = x_1, \end{aligned} \tag{1.2}$$

where $f : J \times R \to R$.

The classical ODE (1.2) has been studied in the literature by several authors for different aspects of the solutions. See for example, Heikkilä and Lakshikantham [9] and the references therein. In this paper, I discuss the RDE (1.1) for existence of random solutions, under suitable conditions of the nonlinearity f which thereby generalize several existence results of the RDE (1.2) proved in the above papers. My analyses on the random of the nonlinear alternative of Leray-Schauder type(Dhage[4, 5] and an algebraic random fixed point theorem of Dhage[4].

Auxiliary Results

Theorem 2.1 (Dhage [4, 5]) Let E be a separable Banach space and let $Q : \Omega \times E \to E$ be a completely continuous random operator. Then, either

1. the random equation $Q(\omega)x = x$ has a random solution, i.e. there is a measurable function $\xi : \Omega \to E$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$, or
2. the set $\varepsilon = \{x : \Omega \to E \; is\, measurable / \lambda(\omega)Q(\omega)x = x\}$ is unbounded for some measurable $\lambda : \Omega \to R$ with $0 < \lambda(\omega) < 1$ on Ω .

An immediate corollary to above theorem in applicable form is

Corollary 2.1 Let E be a separable Banach space and let $Q : \Omega \times E \to E$ be a completely continuous random operator. Then, either

1. the random equation $Q(\omega)x = x$ has a random solution, i.e. there is a measurable function $\xi : \Omega \to E$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$, or
2. theset $\varepsilon = \{x : \Omega \to E \; is\, measurable / \lambda Q(\omega)x = x\}$ is unbounded for some $\omega \in \Omega$ satisfying $0 < \lambda < 1$.

The following theorem is used in the study of nonlinear discontinuous random differential equations.

Theorem 2.2 (Carath'eodory) Let $Q : \Omega \times E \to E$ be a mapping such that $Q(., x)$ is measurable for all $x \in E$ and $Q(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to Q(\omega, x)$ is jointly measurable.

The following lemma is useful in the study of second order initial value problems of ordinary random differential equations via fixed point techniques.

Lemma2.1. For any function $h: J \to L^1(J,R)$, a function $x: J \to C^1(J,R)$ is a solution to the differential equation,

$$-x''(t) = h(t) \quad a.e.\ t \in J,$$
$$x(0) = q_0, x'(0) = q_1, \tag{2.1}$$

if and only if it is a solution of the integral equation.

$$x(t) = q_0 + q_1 t + \int_0^t (t-s)h(s)\,\mathrm{ds}\ . \tag{2.2}$$

Existence Results

Some basic definitions are needed summarise as follows

Definition 3.1 A Function $f: J \times R \times \Omega \to R$ is called random Carath'eodory if the following conditions are satisfied:-

1. the map $(t,\omega) \to (t,x,\omega)$ is jointly measurable for all $x \in R$, and
2. the map $x \to f(t,x,\omega)$ is continuous for all $t \in J$ and $\omega \in \Omega$.

Definition 3.2 A Carath'eodory function $f: J \times R \times \Omega \to R$ is called random L^1-Caratheodory if for each real number r > 0 there is a measurable and bounded function $h_r: \Omega \to L^1(J,R)$ such that

$$|f(t,x,\omega)| \le h_r(t,\omega) \text{ a.e. } t \in J$$

for all $\omega \in \Omega$ and $x \in R$ with $|x| \le r$. Similarly, a Carath'eodory function *f* is called random L^1_R-Carath'eodory if there is a measurable and bounded function $h: \Omega \to L^1(J,R)$ such that

$$|f(t,x,\omega)| \le h(t,\omega) \text{ a.e. } t \in J$$

for all $\omega \in \Omega$ and $x \in R$.

I consider the following set of hypotheses in what follows:

(H_0) The functions $q_0, q_1: \Omega \to R$ are measurable and bounded with

$Q_0 = Sup_{\omega\in\Omega} q_0(\omega)$ and $Q_1 = Sup_{\omega\in\Omega} q_1(\omega)$.

(H_1) The functions *f* is random Carath'eodory on $J \times R \times \Omega$.

(H_2) There exists a measurable and bounded function $\gamma: \Omega \to L^1(J,R)$ and a continuous and nondecreasing function $\gamma: R_+ \to (0,\infty)$ such that

$$\left| f(t,x,\omega) \right| \le \gamma(t,\omega)\psi(|x|) \text{ a. e. } t \in J$$

for all $\omega \in \Omega$ and $x \in R$.Moreover, assume that $\int_C^\infty \frac{dr}{\psi(r)} = \infty$ for all $C \ge 0$.

Main Existence Result

Theorem 3.1 Assume that the hypotheses (H_0) – (H_2) hold. Suppose that

$$\int_C^\infty \frac{dr}{\psi(r)} > \|\gamma(\omega)\|_{L^1} \tag{3.1}$$

for all $\omega \in \Omega$, where $C = Q_0 + Q_1 T$. Then the RDE (1.1) has a random solution defined on J.

Proof . Set $E = C(J,R)$ and define a mapping $Q : \Omega \times E \to E$ by

$$Q(\omega)x(t) = q_0(\omega) + q_1(\omega)t + \int_0^t (t-s)\, f(s, x(s,\omega), \omega)\, ds \tag{3.2}$$

for all $t \in J$ and $\omega \in \Omega$.

Now the map $t \to q_0(\omega) + q_1(\omega)t$ is continuous for all $\omega \in \Omega$. Again, as the indefinite integral is continuous on J, $Q(\omega)$ defines a mapping $Q : \Omega \times E \to E$. we show that Q satisfies all the conditions of Corollary 2.1 on E.

First, I show that Q is random operator on E. Since $f(t,x,\omega)$ is random Carath'eodory, the map $\omega \to f(t,x,\omega)$ is measurable in view of Theorem 2.2 Similarly, the product $(t-s) f(s, x(s,\omega), \omega)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$\omega \to q_0(\omega) + q_1(\omega)t + \int_0^t (t-s)\, f(s, x(s,\omega), \omega)\, ds = Q(\omega)x(t)$$

is measurable. As a result, Q is a random operator on $\Omega \times E$ into E.

Let B be a bounded subset of E. then, there is real number $r > 0$ such that $\|x\| \le r$ for all $x \in B$. Next, I show that the random operator $Q(\omega)$ is continuous on B. let $\{x_n\}$ be a sequence of points in B converging to the point $x \in B$. Then it is enough to prove that $\lim_{n\to\infty} Q(\omega)\, x_n(t) = Q(\omega)\, x(t)$ for all $t \in J$ and $\omega \in \Omega$. By the dominated convergence theorem, then,

$$\lim_{n\to\infty} Q(\omega)\,x_n(t) = q_0(\omega) + q_1(\omega)\,t + \lim_{n\to\infty}\int_0^t (t-s)\,f(s, x_n(s,\omega),\omega)\,ds$$

$$= q_0(\omega) + q_1(\omega)\,t + \int_0^t (t-s)\lim_{n\to\infty}\left[f(s, x_n(s,\omega),\omega)\right]ds$$

$$= q_0(\omega) + q_1(\omega)\,t + \int_0^t (t-s)\left[f(s, x(s,\omega),\omega)\right]ds$$

$$= Q(\omega)x(t)$$

for all $t \in J$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a continuous random operator on E .

Now, I show that $Q(\omega)$ is a totally bounded random operator on E. I prove that $Q(\omega)(B)$ is a totally bounded subset of E for each bounded subset B of E. To finish, it is enough to prove that $Q(\omega)(B)$ is a uniformly bounded and equi-continuous set in E for each $\omega \in \Omega$. Since the map $\omega \to \gamma(t,\omega)$ is bounded, by hypothesis (H_1), there is a constant c such that $\|\gamma(\omega)\|_{L^1} \le c$ for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed. Then for any $x : \Omega \to B$, one has,

$$|Q(\omega)x(t)| \le |q_0(\omega)| + |q_1(\omega)|t + \int_0^t (t-s)\,|f(s, x(s,\omega),\omega)|\,ds$$

$$\le |q_0(\omega)| + |q_1(\omega)|t + \int_0^t (t-s)\,\gamma(s,\omega)\psi(|x(s,\omega)|)\,ds$$

$$\le Q_0 + Q_1\,t + \int_0^T (t-s)\,\gamma(s,\omega)\psi(\|x(\omega)\|)\,ds$$

$$\le (Q_0 + Q_1 T) + T\,\|\gamma(\omega)\|_{L^1}\,\psi(r) \le K_1 ,$$

for all $t \in J$, where $K_1 = (Q_0 + Q_1 T) + cT\psi(r)$. This shows that $Q(\omega)(B)$ is a uniformly bounded subset of E for each $\omega \in \Omega$.

Next, I show that $Q(\omega)(B)$ is an equi-continuous set in E. Let $x \in B$ be arbitrary. Then, for any $t_1, t_2 \in J$, one has

$$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \le |q_1(\omega)t_1 - q_1(\omega)t_2|$$
$$+ \left|\int_0^{t_1} (t_1 - s) f(s, x(s,\omega),\omega)\,ds - \int_0^{t_2} (t_2 - s) f(s, x(s,\omega),\omega)\,ds\right|$$

$$
\begin{aligned}
&\le |q_1(\omega)||t_1-t_2| \\
&\quad + \left| \int_0^{t_1} (t_1-s) f(s,x(s,\omega),\omega)\, ds - \int_0^{t_1} (t_2-s) f(s,x(s,\omega),\omega)\, ds \right| \\
&\quad + \left| \int_0^{t_1} (t_2-s) f(s,x(s,\omega),\omega)\, ds - \int_0^{t_2} (t_2-s) f(s,x(s,\omega),\omega)\, ds \right| \\
&\le |q_1(\omega)||t_1-t_2| \\
&\quad + \left| \int_0^{T} (t_1-t_2) f(s,x(s,\omega),\omega)\, ds \right| + \left| \int_{t_2}^{t_1} (t_2-s) f(s,x(s,\omega),\omega)\, ds \right| \\
&\le Q_1 |t_1-t_2| + \left| \int_0^{T} (t_1-t_2) f(s,x(s,\omega),\omega)\, ds \right| + \left| \int_{t_2}^{t_1} T\, f(s,x(s,\omega),\omega)\, ds \right| \\
&\le Q_1 |t_1-t_2| + \int_0^{T} |t_1-t_2| | f(s,x(s,\omega),\omega)| ds \\
&\quad + \int_{t_2}^{t_1} T\, | f(s,x(s,\omega),\omega)| ds \\
&\le Q_1 |t_1-t_2| + \int_0^{T} |t_1-t_2| \gamma(s,\omega)\psi(|x(s,\omega)|\, ds \\
&\quad + \int_{t_2}^{t_1} T \gamma(s,\omega)\psi(|x(s,\omega)| ds \\
&\le Q_1 |t_1-t_2| + |t_1-t_2| \|\gamma(\omega)\|_{L^1} \psi(r) \\
&\quad + | p(t_1,\omega) - p(t_2,\omega)| \\
&\le [Q_1 + c\psi(r)] |t_1-t_2| + | p(t_1,\omega) - p(t_2,\omega)| \qquad (3.3)
\end{aligned}
$$

for all $\omega \in \Omega$,

where $p(t,\omega) = \int_0^t T\gamma(s,\omega)\psi(r) ds$.

Hence, for all $t_1, t_2 \in J$,

$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \to 0$ as $t_1 \to t_2$, uniformly for all $x \in B$ and $\omega \in \Omega$. Therefore, $Q(\omega)(B)$ is an equi-continuous set in E. As $Q(\omega)(B)$ is uniformly bounded and equi-continuous, it is compact by the Arzela-Ascolli theorem for each $\omega \in \Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on B.

Finally, I prove that the set ε given in conclusion (ii) of corollary 2.1 does not hold. Let $u \in \varepsilon$ be arbitrary and let $\omega \in \Omega$ be fixed. Then $u(t,\omega) = \lambda Q(\omega)u(t)$ for all $t \in J$ and $\omega \in \Omega$, where $0 < \lambda < 1$. Then, one has

$|u(t,\omega)| \le \lambda |Q(\omega)u(t)|$

$$\leq \ |q_0(\omega)| + |q_1(\omega)|t + \int_0^t (t-s)\ |f(s,u(s,\omega),\omega)|ds$$

$$\leq Q_0 + Q_1 T + \int_0^t (t-s)\gamma(s,\omega)\psi(|u(s,\omega)|)ds \leq C + T + \int_0^t \gamma(s,\omega)\psi(|u(s,\omega)|)ds$$

for all $t \in J$ and $\omega \in \Omega$, *where* $C = Q_0 + Q_1 T$.

Let $m(t,\omega) = \sup_{s\in[0,t]} |u(s,\omega)|$. Then, there is a $t^* \in [0,t]$ such that $m(t,\omega) = |u(t^*,\omega)|$. Then from the inequality (3.3) it follows that

$$m(t,\omega) = |u(t^*,\omega)|$$

$$= Q_0 + Q_1 T + T\int_0^{t^*} \gamma(s,\omega)\psi(|u(s,\omega)|)ds$$

$$\leq \ C + T + T\int_0^t \gamma(s,\omega)\psi(m(s,\omega))ds \cdot$$

Put, $$w(t,\omega) = C + T\int_0^t \gamma(s,\omega)\psi(m(s,\omega))ds$$

for $t \in J$. Now differentiating this with respect to t, we obtain

$$w^1(t,\omega) = T\gamma(t,\omega)\psi(m(t,\omega)$$
$$w(0,\omega) = C\,,$$

for $t \in J$. Fom the above inequality, I obtain

$$w^1(t,\omega) \leq T\gamma(t,\omega)\psi(w(t,\omega)\,,$$
$$w(0,\omega) = C\,,$$

or

$$\frac{w^1(t,\omega)}{\psi(w(t,\omega))} \leq T\gamma(t,\omega),$$
$$w(0,\omega) = C.$$

Integrating from 0 to t, we get

$$\int_0^t \frac{w^1(s,\omega)}{\psi(w(s,\omega))}ds \leq T\int_0^t \gamma(s,\omega)\ ds \cdot$$

By change of variable,

$$\int_{C}^{w(t,\omega)} \frac{dr}{\psi(r)} \leq T\,\|\gamma(\omega)\|_{L^1} < \int_{C}^{\infty} \frac{dr}{\psi(r)} = \infty \cdot$$

Now an application of the mean value theorem for integral calculus, there exists a constant $M > 0$ such that

$$u(t,\omega) \leq m(t,\omega) \leq w(t,\omega) \leq M$$

for all $t \in J$ and $\omega \in \Omega$. Hence the condition (ii) of Corollary 2.1 does not hold. As a result, the condition(i) holds and the operator equation $Q(\omega)x = x$ has a random solution. This further implies that the random differential equation(1.1)has a random solution defined on $\Omega \times J$. This completes the proof.

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