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Flow of an Incompressible Couple Stress Fluid in the Region between Two Slowly Rotating Confocal Oblate Spheroids

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Abstract

The paper is aimed at examining the steady flow of an incompressible couple stress fluid in the region bounded by two confocal oblate spheroids when they are slowly rotating about their common axis of symmetry with different angular speeds.

Assuming the angular speeds to be small, we adopt the Stokesian approach and neglect the non linear terms in the equations of motion. The velocity is determined subject to the hyper stick boundary condition. The expression for velocity is obtained in terms of associated Legendre functions and oblate spheroidal radial and angular wave functions. However, for convenience, we write the expression of velocity in terms of prolate spheroidal radial and angular wave functions, in addition to the associated Legendre functions. The couple acting on the two spheroids is obtained. The variation of the couple is studied numerically for different values of the couple stress parameter, rotation parameter and geometric parameter. The results for the non polar case are also presented for completeness. The analysis of the problem is facilitated by the introduction of a certain notations.

Keywords: Couple stress fluids; Confocal oblate spheroid; Stokesian approach; Angular and radial spheroidal wave functions; Velocity; Couple.

Introduction

The theory of couple stress fluids was initiated by V.K. Stokes [1, 2] almost parallel to the theory of micropolar fluids of A.C. Eringen [3]. The animal blood is one of the examples close to the two fluid models. These two prominent polar fluid models are independent generalizations to the classical viscous fluid model which arise from different stand points. The couple stress fluid model of Stokes takes into consideration the mechanical interactions taking place across a surface in the fluid medium and is not concerned with the micro structure as is the case with micropolar fluid. This model is the simplest polar fluid model that shows all the important features and effects of couple stresses. The characterizing features that distinguish the couple stress fluid from the classical Newtonian fluid are the presence of couple stresses and body couples in the medium and the non symmetry of the stress tensor. The governing equations of the couple stress fluid flow are similar to the classical Navier Stokes Equations with an increase in the order of the equation by two. The structure of the equations facilitates a comparison with the results for the classical Newtonian fluid which is nonpolar.

A good number of fluid flow problems that are present in the realm of viscous fluid theory have also been investigated in the context of the couple stress fluid theory during the last four and half decades. Stokes himself has studied the effects of couple stresses in fluids on the creeping flow past a sphere [4]. He also studied the effects of couple stresses on hydromagnetic channel flows. A number of references to couple stress fluid flow problems dealt with up to 1983 can be seen in Stokes [2]. Lakshmana Rao and Iyengar made analytical and computational studies of the couple stress fluid flows with respect to certain axisymmetric bodies like circular cylinder, sphere and spheroid in [5]. Ramkissoon obtained a formula for the drag on a general axisymmetric body when there is a uniform flow of an incompressible couple stress fluid far away from the body parallel to the axis of symmetry [6] which is analogous to the elegant formula derived by Payne and Pell [7]. Subsequently he also obtained a formula in the form of a limit for the couple acting on a general axi symmetric body rotating slowly in an infinite expanse of an incompressible couple stress fluid [8] and this is analogous to the one derived by Kanwal for the case of a viscous fluid [9]. Iyengar and Srinivasacharya studied the Stokes flow of a couple stress fluid past an approximate sphere [10] and the couple experienced by an approximate sphere in an incompressible couple stress fluid [11].

There have been some studies of couple stress fluid flow problems in other contexts too. Dabe and Mohandis studied the effect of couple stresses on pulsatile hydro magnetic Poiseuille flow [12]. Naduvinamani et.al studied the squeeze film lubrication of short porous general bearing with couple stress fluids[13]. They also discussed the surface roughness effects in a short porous journal bearing [14] and the effects of surface roughness on the couple stress squeeze film between sphere and a flat plate [15]. Naduvinamani et. al studied the hydrodynamic lubrication of rough slider bearings with couple stress fluids [16]. Rathod and Tanveer studied Pulsatile flow of Couple stress fluid through a porous medium with periodic body acceleration and magnetic field [17]. Devakar and Iyengar studied Stokes problems and run uip flow between parallel plates for an incompressible couple stress fluid [18, 19]. Iyengar and Punnamchader discussed the pulsating flow of a couple stress fluid between permeable beds with an imposed magnetic field [20]. These diverse flow problems indicate the continuing interest among researchers in couple stress fluid flows.

In this paper, we consider two confocal oblate spheroids the region between which is filled by an incompressible couple stress fluid. We discuss the steady flow of the fluid generated by the slow steady rotation of the two spheroids about their common axis of symmetry with two different angular speeds. This problem in the case of two concentric spheres can be handled easily. The problem under consideration requires the solution of a fourth order partial differential equation which has to satisfy the boundary conditions on the two spheroids. Thus the problem is mathematically difficult and challenging in view of the fourth order partial differential equation governing the flow and the spheroidal geometry. We obtain the velocity of the flow field and an expression for the couple experienced by the bounding spheroids. Numerical evaluation of the couple is made for different values of the couple stress, rotation and geometric parameters and its variation is studied. The case of viscous fluid is also included for completeness. As the situation demanded, we have introduced some notations conveniently to handle lengthy expressions. The problem under consideration is significant in view of the mathematical complications that it poses and its tractability using special functions like the angular and radial spheroidal wave functions. The authors feel that the notations that are introduced herein in can be useful for future workers.

Basic Equations and Formulation of the Problem

Let the region between two confocal oblate spheroids be filled by an incompressible couple stress fluid. Let their common axis of symmetry be taken as z-axis and the common center be taken as the origin. Let us introduce an oblate spheroidal coordinates (ξ, η, ϕ) through the usual cylindrical polar coordinate system (r, z, ϕ) such that

$$z + ir = c \text{ Sin h}(\xi + i\eta) \tag{1}$$

and let

$$\text{Sinh } \xi = T, \text{Cos } \eta = t \tag{2}$$

Let the inner and outer spheroids be defined by $\xi = \xi_0$ and $\xi = \xi_1$ respectively. Equivalently let these be represented by $T = T_0$ and $T = T_1$. Let these rotate with angular speeds Ω_0 and Ω_1 about the common axis. We assume that Ω_0 and Ω_1 are so small that the equations of motion can be linearized by the neglect of nonlinear terms in the equation of motion under the Stokesian approach. The linearized version of the equations of motion is given by

$$\text{div } \bar{q} = 0 \tag{3}$$

$$\rho \frac{d\bar{q}}{dt} = -\text{ grad p} - \mu \text{ curl curl } \bar{q} - \eta_1 \text{ curl curl curl curl } \bar{q} = 0 \tag{4}$$

Since Ω_0 and Ω_1 are assumed to be small we can assume that the velocity $\overline{q}$ has only the toroidal component in the direction of the unit vector $\overline{e_\phi}$. The flow variables are all assumed to be independent of ϕ. Hence we can choose the velocity $\overline{q}$ as,

$$\overline{q} = V(\xi, \eta)\,\overline{e_\phi} \tag{5}$$

Using (5) in (4), we note that V (ξ, η) satisfies the equation

$$E^2\left(E^2 - \frac{\lambda^2}{c^2}\right)(h_3 V) = 0 \tag{6}$$

where

$$\frac{\lambda^2}{c^2} = \frac{\mu}{\eta_1} \tag{7}$$

The above V can be treated as a function of T and t and the operator E^2 is given by

$$E^2 = \frac{1}{c^2\left(T^2 + t^2\right)}\left[\left(T^2 + 1\right)\frac{\partial^2}{\partial T^2} + \left(1 - t^2\right)\frac{\partial^2}{\partial t^2}\right] \tag{8}$$

The velocity component V(T, t) has to satisfy the hyperstick boundary condition on $T=T_0$ and $T=T_1$ and in view of this, the velocity must satisfy the following four conditions:

$$V(T_0, t) = \Omega_0\, c\,\sqrt{\left(T_0^2 + 1\right)\left(1 - t^2\right)} \tag{9}$$

$$V(T_1, t) = \Omega_1\, c\,\sqrt{\left(T_1^2 + 1\right)\left(1 - t^2\right)} \tag{10}$$

$$\left(\frac{\partial V}{\partial T}\right)_{T_0} = \frac{\Omega_0\, c\, T_0\sqrt{\left(1 - t^2\right)}}{\sqrt{\left(T_0^2 - 1\right)}} \tag{11}$$

$$\left(\frac{\partial V}{\partial T}\right)_{T_1} = \frac{\Omega_1\, c\, T_1\sqrt{\left(1 - t^2\right)}}{\sqrt{\left(T_1^2 - 1\right)}} \tag{12}$$

Stokes in [2] mentions two types of boundary conditions: (A) Absence of couple stresses on the boundary; (B) Vorticity at the boundary equals the rate of rotation of the boundary. These are in addition to the usual no-slip condition on the velocity at the boundary. The conditions (9) and (10) correspond to the no-slip condition and conditions (11) and (12) correspond to the condition (B) mentioned above. Thus the velocity V can be determined by solving the equation (6) subject to the boundary conditions (9), (10), (11) and (12).

Solution of the Problem

The solution of (6) can be obtained in the form $V = V_1 + V_2$ by superposing the solutions of

$$E^2 (h_3 V_1) = 0 \tag{13}$$

and

$$\left(E^2 - \frac{\lambda^2}{c^2}\right)(h_3 V_2) = 0 \tag{14}$$

in view of the linearity of the differential operators E^2 and $\left(E^2 - \frac{\lambda^2}{c^2}\right)$ and their commutativity. Adopting the method of separation of variables, the solution of (13) which is regular on the axis is given by

$$V_1(T, t) = \sum_{n=1}^{\infty}\left[A_n\, P_n^{(1)}(iT) + B_n\, Q_n^{(1)}(iT)\right] P_n^{(1)}(t) \tag{15}$$

where $P_n^{(1)}$, $Q_n^{(1)}$ are associated Legendre functions. The solution of (14) can be obtained in terms of oblate spheroidal wave functions $R_{1n}(i\lambda, T)$ and $S_{1n}(i\lambda, t)$. The oblate spheroidal angular wave functions can also be represented as prolate spheroidal angular wave functions by changing the parameter $i\lambda$ to λ. The radial oblate spheroidal wave functions can be represented as radial prolate spheroidal wave functions by changing the parameter $i\lambda$ to λ and the variable T to iT [21]. Hence, the solution V_2 of the equation (14) which is regular on the axis is given by

$$V_2(T, t) = \sum_{n=1}^{\infty}\left[C_n\, R_{1n}^{(3)}(\lambda, iT) + D_n R_{1n}^{(4)}(\lambda, iT)\right] S_{1n}^{(1)}(\lambda\, t) \tag{16}$$

where C_n and D_n are infinite sets of arbitrary constants and the functions $R_{1n}^{(3)}$, $R_{1n}^{(4)}$ and $S_{1n}^{(1)}$ are prolate spheroidal wave functions given by

$$R_{1n}^{(3)}(\lambda\, , iT) = \left\{i^{n+2} \sum_{r=0,1}^{\infty\prime}(r+1)(r+2)d_r^{1n}(\lambda)\right\}^{-1} \left(\frac{2(T^2+1)}{\pi\lambda T^3}\right)^{1/2}$$

$$\sum_{r=0,1}^{\infty\prime} (r+1)(r+2)d_r^{1n}(\lambda)\; K_{r+(3/2)}(\lambda T) \tag{17}$$

$$R_{1n}^{(4)}(\lambda, iT) = \left\{i^{n-1} \sum_{r=0,1}^{\infty\prime}(r+1)(r+2)d_r^{1n}(\lambda)\right\}^{-1} \left(\frac{2(T^2+1)}{\pi\lambda T^3}\right)^{1/2}$$

$$\sum_{r=0,1}^{\infty'} (-1)^{r+1}(r+1)(r+2) d_r^{1n}(\lambda)\ K_{r+\left(3/2\right)}(-\lambda T)\ (18)$$

and

$$S_{1n}^{(1)}(\lambda,\ t) = \sum_{r=0,1}^{\infty'} d_r^{1n}(\lambda)\ P^{(1)}{}_{r+1}(t) \tag{19}$$

Hence the velocity component V is given by
V (T, t) = V_1 (T, t) + V_2 (T, t)

$$= \sum \left[A_n\, P_n^{(1)}(iT) + B_n\, Q_n^{(1)}(iT)\right] P_n^{(1)}(t)$$

$$+ \sum \left[C_n\, R_{1n}^{(3)}(i\lambda, s) + D_n R_{1n}^{(4)}(i\lambda, s)\right] S_{1n}^{(1)}(i\lambda, t) \tag{20}$$

The infinite sets of arbitrary constants A_n, B_n, C_n, D_n are to be determined making use of the boundary conditions (9), (10), (11) and (12). In view of symmetry, $V(T, t) = V(T, -t)$, the constants { A_n, B_n, C_n, D_n } can all be presumed to be zero for even values of n.

Determination of Arbitrary Constants

The arbitrary constants A_n, B_n, C_n, D_n can be determined, using the expression of velocity given in (20) in the boundary conditions (9), (10), (11) and (12). Implementation of the boundary conditions results in the following equations

$$\sum_{n=1}^{\infty}\left[A_n\, P_n^{(1)}(i\,T_0) + B_n\, Q_n^{(1)}(i\,T_0)\right] P_n^{(1)}(t)$$

$$+ \sum_{n=1}^{\infty} \left[C_n\, R_{1n}^{(3)}(\lambda, iT_0) + D_n R_{1n}^{(4)}(\lambda, i\,T_0)\right] S_{1n}^{(1)}(\lambda, t)$$

$$= \Omega_0\ c\ \sqrt{\left(T_0^2+1\right)\left(1-t^2\right)} \tag{21}$$

$$\sum_{n=1}^{\infty}\left[A_n\, P_n^{(1)}(iT_1) + B_n\, Q_n^{(1)}(iT_1)\right] P_n^{(1)}(t)$$

$$+ \sum_{n=1}^{\infty} \left[C_n\, R_{1n}^{(3)}(\lambda, iT_1) + D_n R_{1n}^{(4)}(\lambda, iT_1)\right] S_{1n}^{(1)}(\lambda, t)$$

$$= \Omega_1\ c\ \sqrt{\left(T_0^2+1\right)\left(1-t^2\right)} \tag{22}$$

$$\sum_{n=1}^{\infty}\left[A_n \frac{d}{dT_0}\left(P_n^{(1)}(iT_0)\right)+B_n \frac{d}{dT_0}\left(Q_n^{(1)}(iT_0)\right)\right]P_n^{(1)}(t)$$

$$+\sum_{n=1}^{\infty}\left[C_n \frac{d}{dT_0}\left(R_{1n}^{(3)}(\lambda, iT_0)\right)\right.$$

$$\left.+D_n \frac{d}{dT_0}\left(R_{1n}^{(4)}(\lambda, iT_0)\right)\right]S_{1n}^{(1)}(\lambda, t)=\frac{\Omega_0 \, c\, T_0\sqrt{\left(1-t^2\right)}}{\sqrt{\left(T_0^2+1\right)}} \qquad (23)$$

$$\sum_{n=1}^{\infty}\left[A_n \frac{d}{dT_1}\left(P_n^{(1)}(iT_1)\right)+B_n \frac{d}{dT_1}\left(Q_n^{(1)}(iT_1)\right)\right]P_n^{(1)}(t)$$

$$+\sum_{n=1}^{\infty}\left[C_n \frac{d}{dT_1}\left(R_{1n}^{(3)}(\lambda, iT_1)\right)+D_n \frac{d}{dT_1}\left(R_{1n}^{(4)}(\lambda, iT_1)\right)\right]S_{1n}^{(1)}(\lambda, t)$$

$$=\frac{\Omega_1 \, c\, T_1\sqrt{\left(1-t^2\right)}}{\sqrt{\left(T_1^2+1\right)}} \qquad (24)$$

Multiplying equations (21) and (22) with $P_n^{(1)}(t)$ and integrating both sides with respect to t from –1 to 1. We get

$$\left[A_m P_m^{(1)}(iT_0)+B_m Q_m^{(1)}(iT_0)\right]$$

$$+\sum_{l}\left[C_l R_{1l}^{(3)}(\lambda, iT_0)+D_l R_{1l}^{(4)}(\lambda, iT_0)\right] d_{m-1}^{1l}(\lambda)$$

$$=\Omega_0 c\sqrt{\left(T_0^2+1\right)}\delta_{m_1} \quad m = 1, 3, 5\ldots\ldots \qquad (25)$$

$$\left[A_m P_m^{(1)}(iT_1)+B_m Q_m^{(1)}(iT_1)\right]$$

$$+\sum_{l}\left[C_l R_{1l}^{(3)}(\lambda, iT_1)+D_l R_{1l}^{(4)}(\lambda, iT_1)\right] d_{m-1}^{1l}(\lambda)$$

$$=\Omega_1 c\sqrt{\left(T_1^2+1\right)}\delta_{m_1} \quad m = 1, 3, 5\ldots \qquad (26)$$

Similarly, multiplying the equations (23) and (24) with $P_n^{(1)}(t)$ and integrating with respect to t from –1 to 1, we get

$$\left[A_m DP_m^{(1)}(iT_0)+B_m DQ_m^{(1)}(iT_0)\right]$$

$$+\sum_{n=1}^{\infty}\left[C_l DR_{1l}^{(3)}(\lambda, iT_0)+D_l DR_{1l}^{(4)}(\lambda, iT_0)\right] d_{m-1}^{1l}(\lambda)$$

$$=\frac{\Omega_0 \, c\, T_0}{\sqrt{\left(T_0^2+1\right)}}\,\delta_{m1} \qquad (27)$$

$$\left[A_m \, DP_m^{(1)}(iT_1) + B_m \, DQ_m^{(1)}(iT_1)\right]$$

$$+ \sum_{n=1}^{\infty} \left[C_l \, DR_{1l}^{(3)}(i\lambda, iT_1) + D_l \, DR_{1l}^{(4)}(\lambda, iT_1)\right] d_{m-1}^{1l}(\lambda)$$

$$= \frac{\Omega_1 \, c \, T_1}{\sqrt{\left(T_1^2 + 1\right)}} \, \delta_{m1} \tag{28}$$

where

$$\frac{d}{dT}\left(P_n^{(1)}(iT)\right) = D\,P_n^{(1)}(iT); \; \frac{d}{ds}\left(R_{il}^{(3)}(\lambda, iT)\right) = D\,R_{il}^{(3)}(\lambda, iT)$$

$$\frac{d}{dT}\left(Q_n^{(1)}(iT)\right) = D\,Q_n^{(1)}(iT); \; \frac{d}{dT}\left(R_{il}^{(4)}(\lambda, iT)\right) = D\,R_{il}^{(4)}(\lambda, iT) \tag{29}$$

Eliminating A_m from equation (25) and equation (26), we get

$$B_m\left[P_m^{(1)}(iT_0)\; Q_m^{(1)}(iT_1) - P_m^{(1)}(iT_1)\; Q_m^{(1)}(iT_0)\right]$$

$$+ \sum_{l} \Big\{ C_l \left[P_m^{(1)}(iT_0) R_{1l}^{(3)}(\lambda, iT_1) - P_m^{(1)}(iT_1) R_{1l}^{(3)}(\lambda, iT_0) \right]$$

$$- P_m^{(1)}(iT_1) R_{1l}^{(4)}(\lambda, iT_0)\Big]\Big\} \, d_{m-1}^{1l}(\lambda) = \Big[P_1^{(1)}(iT_0)\,\Omega_1\, C\sqrt{(T_1^2+1)}$$

$$- P_1^{(1)}(iT_1)\,\Omega_0\, C\sqrt{(T_0^2+1)}\Big] \, \delta_{m1} \tag{30}$$

Eliminating A_m from equations (27) and (28), we get

$$B_m\left[DP_m^{(1)}(iT_0)\, DQ_m^{(1)}(iT_1) - DP_m^{(1)}(iT_1)\; DQ_m^{(1)}(iT_0)\right]$$

$$+ \sum_{l=1}^{\infty} \Big\{ C_l \left[DP_m^{(1)}(iT_0)\, DR_{1l}^{(3)}(\lambda, iT_1) - DP_m^{(1)}(iT_1)\, DR_{1l}^{(3)}(\lambda, iT_0)\right]$$

$$+ D_l \left[DP_m^{(1)}(iT_0)\, DR_{1l}^{(4)}(\lambda, iT_1) - DP_m^{(1)}(iT_1)\, DR_{1l}^{(4)}(\lambda, iT_0)\right]\Big\}$$

$$d_{m-1}^{1l}(\lambda) = \left[\frac{\Omega_1 \, C\, iT_1}{\sqrt{(T_1^2+1)}} DP_1^{(1)}(iT_0) \right.$$

$$\left. - \frac{\Omega_0 \, C \, T_0}{\sqrt{(T_0^2+1)}} DP_1^{(1)}(iT_1)\,\delta_{m1} \right] \tag{31}$$

Let us introduce

$$PQ_m(T_0, T_1) = P_m^{(1)}(iT_0)\; Q_m^{(1)}(iT_1) - P_m^{(1)}(iT_1)\; Q_m^{(1)}(iT_0) \tag{32}$$

$$DP\,DQ_m\,(T_0, T_1) = DP_m^{(1)}(iT_0)\,DQ_m^{(1)}(iT_1) - DP_m^{(1)}(iT_1)\,DQ_m^{(1)}(iT_0) \tag{33}$$

$$P_m^{(1)}\,R_{1l}^{(3)}(T_0, T_1) = P_m^{(1)}(iT_0)R_{1l}^{(3)}(\lambda, iT_1) - P_m^{(1)}(iT_1)R_{1l}^{(3)}(\lambda, iT_0) \tag{34}$$

$$P_m^{(1)}\,R_{1l}^{(4)}(T_0, T_1) = P_m^{(1)}(iT_0)R_{1l}^{(4)}(\lambda, iT_1) - P_m^{(1)}(iT_1)R_{1l}^{(4)}(\lambda, iT_0) \tag{35}$$

$$DP_m^{(1)}\,DR_{1l}^{(3)}(T_0, T_1) = DP_m^{(1)}(iT_0)\,DR_{1l}^{(3)}(\lambda, iT_1) - DP_m^{(1)}(iT_1)\,DR_{1l}^{(3)}(\lambda, iT_0) \tag{36}$$

$$DP_m^{(1)}\,DR_{1l}^{(4)}(T_0, T_1) = DP_m^{(1)}(iT_0)\,DR_{1l}^{(4)}(\lambda, iT_1) - DP_m^{(1)}(iT_1)\,DR_{1l}^{(4)}(\lambda, iT_0) \tag{37}$$

With this notation, rewriting the equations (30) and (31) and then eliminating B_m, we get the following infinite simultaneous nonhomogeneous system of linear equations in C_l and D_l:

$$\sum_{l=1}^{\infty} \left(A_{ml}\,C_l + B_{ml}\,D_l\right) = \Delta_m \tag{38}$$

where

$$A_{ml} = \Big[\ DP\,DQ_m\,(T_0, T_1)\ P_m^{(1)}\,R_{1l}^{(3)}(T_0, T_1)$$

$$- P\,Q_m\ (T_0, T_1)\ DP_m^{(1)}\,DR_{1l}^{(3)}(T_0, T_1)\ \Big]\ d_{m-1}^{1l}(\lambda) \tag{39}$$

$$B_{ml} = \Big[\ DP\,DQ_m\,(T_0, T_1)\ P_m^{(1)}\,R_{1l}^{(4)}(T_0, T_1)$$

$$- P\,Q_m\ (T_0, T_1)\ DP_m^{(1)}\,DR_{1l}^{(4)}(T_0, T_1)\ \Big]\ d_{m-1}^{1l}(\lambda) \tag{40}$$

and

$$\Delta_1 = ic\left(\Omega_1 - \Omega_0\right)\Big[\sqrt{\left(T_0^2+1\right)\left(T_1^2+1\right)}\ DP\,D\,Q_1\,(T_0, T_1)$$

$$- \frac{T_0\,T_1}{\sqrt{\left(T_0^2+1\right)\left(T_1^2+1\right)}}\,PQ_1\,(T_0, T_1)\ \Big] \tag{41}$$

and

$$\Delta_m = 0 \ \text{ for } \quad m \geq 2$$

As the system in (38) is infinite, the constants C_l and D_l can be determined by a numerical procedure after fixing a suitable stage of truncation. After the determination of C_l and D_l, B_m's can be determined making use of either of (30) and (31) subsequently A_m's can be determined using any one of (27) and (28). Thus we have a systematic feasible procedure for the numerical determination of A_n, B_n, C_n, D_n. Here an explicit analytical determination of the constants is not possible.

Couple Acting on the Spheroids

The couple acting on the two spheroids $T=T_0$ and $T=T_1$ can be evaluated by calculating the contributions due to the action of the force stress tensor and the couple stress tensor. The contribution of the force stress tensor is given by

$$C_I = 2\pi c^3 (T^2+1) \int_{-1}^{1} \sqrt{(T^2+t^2)(1-t^2)}\, \left(t_{\xi\phi}\right)_T \, dt \tag{42}$$

$$= 2\pi c^3 (T^2+1) \int_{-1}^{1} \left(-\frac{1}{2c}(1-t^2)\frac{\partial m}{\partial t}\right) dt$$

$$-\frac{\mu}{c} \int_{-1}^{1} \left[\left(\sqrt{(T^2+1)}\frac{\partial V_2}{\partial T} + \frac{T}{\sqrt{(T^2+1)}} V_2\right) \sqrt{(1-t^2)}\, dt\right] \tag{43}$$

where the integrand is to be evaluated on $T = T_0$ or $T = T_1$ as the case may be. This, on evaluation turns out to be

$$C_I = - \quad 2\pi c^3 (T^2+1) \int_{-1}^{1} t\, m \, dt$$

$$-\frac{8\pi}{3} \mu c^2 \left(T^2+1\right)^{3/2} \sum \left(C_n DR_{1n}^{(3)}(\lambda, iT) + D_n DR_{1n}^{(4)}(\lambda, iT)\right)$$

$$d_0^{1n}(\lambda) + \frac{8\pi}{3} \mu c^2\, T \left(T^2+1\right)^{1/2} \sum \left(C_n DR_{1n}^{(3)}(\lambda, iT)\right.$$

$$\left. + D_n DR_{1n}^{(4)}(\lambda, iT)\right) d_0^{1n}(\lambda) \tag{44}$$

on $T = T_0$ and $T = T_1$. The contribution of the couple stress tensor to the couple on the boundary is given by

$$C_{II} \qquad = 2\pi c^2 \sqrt{(T^2+1)} \int_{-1}^{1} \left[t\sqrt{(T^2+1)}\; m_{\xi\xi} - T\sqrt{(1-t^2)}\; m_{\xi\eta}\right] dt$$

$$= 2\pi c^2 \sqrt{(T^2+1)} \int_{-1}^{1} t m(s,t)\, dt$$

$$+ 4\pi T\sqrt{(T^2+1)}\, \eta_1 \lambda^2 \int_{-1}^{1} \sqrt{(1-t^2)}\; V_2(T,t)\, dt$$

$$= 2\pi c^2 (T^2+1) \int_{-1}^{1} t m(T,t)\, dt + \frac{16\pi}{3}\; \eta_1 \lambda^2 T\sqrt{(T^2+1)}$$

$$\sum\left(C_n R_{1n}^{(3)}(\lambda, it) + D_n R_{1n}^{(4)}(\lambda, it)\right) d_0^{1n}(\lambda) \tag{45}$$

The total couple on the body is given by

$$\begin{aligned} C &= C_I + C_{II} \\ &= \frac{8\pi}{3}\mu c^2 \left(T^2+1\right)^{1/2}\Big[-(T^2+1)\sum\left(C_n DR_{1n}^{(3)}(\lambda, iT)\right. \\ &\quad \left. + D_n DR_{1n}^{(4)}(\lambda, iT)\right) d_0^{1n}(\lambda) \\ &\quad + T\sum\left(C_n DR_{1n}^{(3)}(\lambda, iT) + D_n DR_{1n}^{(4)}(\lambda, iT)\right) d_0^{1n}(\lambda)\ \Big] \end{aligned} \tag{46}$$

To calculate the couple C on the inner spheroid $T = T_0$, using (25) and (27) we get

$$\begin{aligned} \sum \left(C_n R_{1n}^{(3)}(\lambda, iT_0) + D_n R_{1n}^{(4)}(\lambda, iT_0)\right) d_0^{1n}(\lambda) &= \Omega_0\, c\sqrt{(T^2+1)} \\ &\quad - \left[A_1 P_1^{(1)}(iT_0) + B_1 Q_1^{(1)}(iT_0)\right] \end{aligned} \tag{47}$$

and

$$\begin{aligned} \sum \left(C_n DR_{1n}^{(3)}(\lambda, iT_0) + D_n DR_{1n}^{(4)}(\lambda, iT_0)\right) d_0^{1n}(i\lambda) &= \frac{\Omega_0\, cT_0}{\sqrt{(T_0^2+1)}} \\ &\quad - \left[A_1 DP_1^{(1)}(iT_0) + B_1 DQ_1^{(1)}(iT_0)\right] \end{aligned} \tag{48}$$

using (47), (48) in (46) and simplifying, we observe that, on $T = T_0$, the couple is given by

$$C = -\frac{16\pi}{3}\mu c^2 B_1 \tag{49}$$

Using (26) and (28) for $T=T_1$ in (46), we see that the couple on $T = T_1$ is also given by

$$C = -\frac{16\pi}{3}\mu c^2 B_1 \tag{50}$$

Thus both the spheroids experience the same couple.

In the case of nonpolar fluid, this expression for couple is same as above, however, with B_1 taken as the one corresponding to the nonpolar case. This formula is the same as that we get in the case of slow steady rotation of a single oblate spheroid in an infinite expanse of an incompressible couple stress fluid which is otherwise at rest as can be seen from [5].

Case of non polar fluid

In this case the equations of motion are

$$\text{div } \bar{q} = 0 \tag{51}$$

$$-\nabla p - \mu \nabla \times (\nabla \times \bar{q}) = 0 \tag{52}$$

and these can be obtained as a special case from couple stress fluid flow equations by putting $\eta_1 = 0$. With $\bar{q} = (0, 0, V(\xi, \eta))$, we get

$$E^2 (h_3 V) = 0 \tag{53}$$

In the present case of rotation of two confocal oblate spheroids, we get

$$V(T, t) = \sum_{n=1}^{\infty} \left[A_n \; P_n^{(1)}(iT) + B_n \; Q_n^{(1)}(iT) \right] \; P_n^{(1)}(t) \tag{54}$$

On the boundary $T = T_0$ and $T = T_1$, we require

$$V(T_0, t) = \Omega_0 c \sqrt{(T_0^2 + 1)(1 - t^2)} \tag{55}$$

$$V(T_1, t) = \Omega_1 c \sqrt{(T_1^2 + 1)(1 - t^2)} \tag{56}$$

With this we get $A_n = B_n = 0$ for $n \geq 2$
and

$$A_1 = c \frac{\left(\Omega_0 \sqrt{(T_0^2 + 1)} \; Q_1^{(1)}(i T_1) - \Omega_1 \sqrt{(T_1^2 + 1)} \; Q_1^{(1)}(i T_0) \right)}{\text{Den}^{(1)}(T_0, T_1)} \tag{57}$$

$$B_1 = -c \frac{\left(\Omega_0 \sqrt{(T_0^2 + 1)} \; P_1^{(1)}(i T_1) - \Omega_1 \sqrt{(T_1^2 + 1)} \; P_1^{(1)}(i T_0) \right)}{\text{Den}^{(1)}(T_0, T_1)} \tag{58}$$

The couple on the on the boundary $T = T_0$ or $T = T_1$ is seen to be

$$-\frac{16}{3} \pi \mu c^2 B_1 \tag{59}$$

which is same as that in (49) with $\eta_1 = 0$. The variation of the couple on the boundary is calculated for varying values of $\Omega = \Omega_1 / \Omega_0$

Numerical Work

To evaluate the couple on the spheroid numerically, we need to evaluate only the coefficient B_1. For this we have to solve the system of equations (39). For a series of fluid flow parameter values, we have solved the system (39) numerically truncating it to a 10 by 10 system. This truncation was motivated by the availability of the quantities $d_r^{mn}(i\lambda)$, $d_r^{mn}(ip)$ only to a limited extent in the published literature [21]. After numerically evaluating C_l, D_l for l = 1, 3, 5, 7, 9, we can numerically compute B_1 from equation (30) or (31).

The couple on the spheroid(s) is calculated for diverse values of the couple stress parameter λ geometric parameter T_1 (for a given T_0) and the rotation parameter Ω. For a given T_0 and T_1, the figures (1) to (4) display the variation of the couple with respect to λ for different values of the rotation parameter Ω. As λ increases, for a Ω, the couple decreases. Further for the same λ, as Ω increases, the couple increases. This trend is seen in all the calculated values.

We evaluate the couple parameter B_1 for various values of the couple stress parameter λ, rotation parameter $\Omega = \Omega_1/\Omega_0$, and geometric parameter T_1 for a fixed inner spheroid size T_0. The results are shown through figures 1 to 4. In all the figures we have taken T_0 = 1.1 and increased T_1 which means that the size of the outer spheroid is increasing. In each figure we have shown the variation of the couple parameter for four values of Ω and for increasing values of λ. Further the couple has a tendency to decrease. Further as Ω increases the couple increases. This trend is observed in all the cases.

In all the cases, we notice that as λ increases, the couple decreases. An increase in λ implies a decrease in the coefficient of couple stress viscosity. This in turn implies that the fluid is tending to be Newtonian. Thus the couple acting on each spheroid in the case of couple stress fluid is greater than the couple experienced in the case of a Newtonian fluid.

From the figures 1 to 4, it is observed that as the size of the outer spheroid increases, the couple is increasing. Further we notice that as the rotation parameter Ω increases, the couple is increasing.

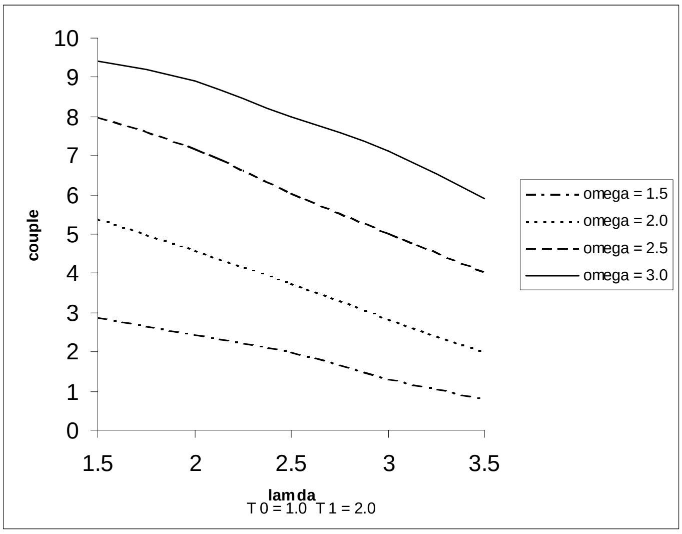

Figure 1: Variation of couple w.r.t lamda with varying omega

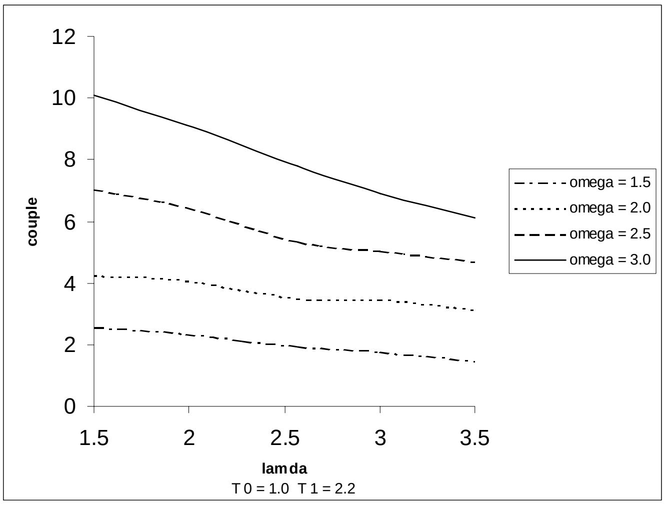

Figure 2: Variation of couple w.r.t lamda with varying omega

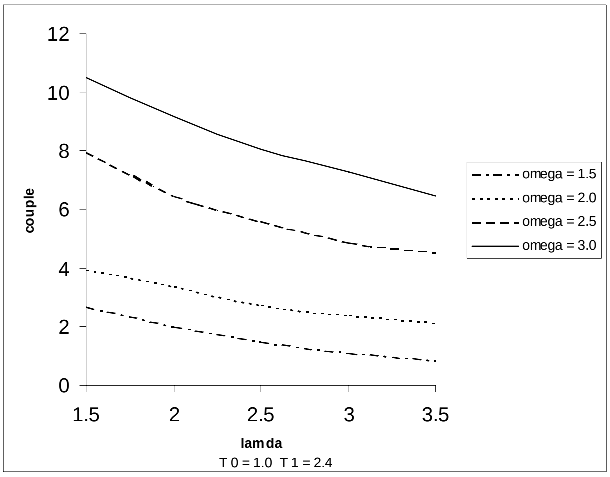

Figure 3: Variation of couple w.r.t lamda with varying omega

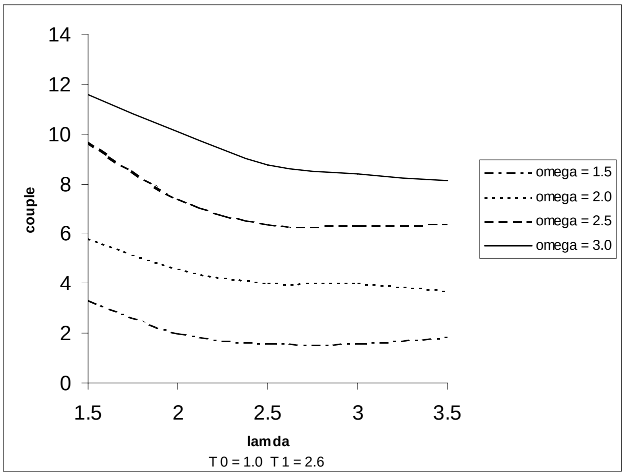

Figure 4: Variation of couple w.r.t lamda with varying omega

Conclusions

In this paper we have considered two confocal oblate spheroids and assumed that the region between them is filled by a couple stress fluid. With the assumption that the spheroids are rotating steadily and slowly with different angular speeds, we obtained expressions for the toroidal velocity component under Stokesian approximation. We obtained an expression for the couple experienced by each of the boundaries. Further we presented the case of Newtonian fluid is for completeness. The analysis is mainly theoretical in nature and the complicated expressions involving spheroidal wave functions could be handled to lead to the determination of the velocity and couple through the introduction of certain notations which are likely to be useful for the workers in the field. Computational work to find the variation of couple is carried out and the variation is presented through graphs. It is found that as the rotation parameter/ the coefficient of couple stress viscosity/ size of the spheroid (when the inner spheroid is fixed) increases the couple experienced by the spheroids is seen to increase.

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