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# t-Pebbling Number of Some Multipartite Graphs

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**Abstract**

Given a configuration of pebbles on the vertices of a graph $G$, a *pebbling* move consists of taking two pebbles off some vertex v and putting one of them back on a vertex adjacent to $v$. A graph is called *pebbleable* if for each vertex $v$ there is a sequence of pebbling moves that would place at least one pebble on $v$. The *pebbling number* of a graph $G$, is the smallest integer $m$ such that $G$ is pebbleable for every *configuration* of $m$ pebbles on $G$. The concept of $t$-pebbling in graphs is the general formulation of pebbling in graphs. In this chapter we characterise the t - pebbling number of a large class of diameter two graphs. We have also established some bounds for the $t$-pebbling number of graphs.

**AMS subject classification:**
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## Introduction

**Pebbling:** F R K Chung [2] introduced the concept of pebbling in graphs. A pebbling configuration $C$ of a graph $G$ is a distribution of pebbles on $G$. A pebbling move consists of removing two pebbles lying on the same vertex $v$ and placing one of these pebbles on some vertex that is adjacent to $v$. For a given pebbling configuration $C$, a vertex $v$ is called pebbleable if there is a sequence of pebbling moves such that at least one pebble can be placed on $v$. The pebbling number of a graph $G$ is the minimum number $m$ of pebbles that ensure that every vertex of $G$ is pebbleable, no matter what initial configuration of $m$ pebbles we start with. Let $f(G, v)$ denotes the pebbling number of a vertex $v$ of $G$, and $f(G)$ the pebbling number of $G$. Hence,

$$f(G) = \max_{v \in V(G)} f(G, v)$$

**t-pebbling:** The concept of $t$-pebbling in graphs is the general formulation of pebbling in graphs. Let $G$ be a connected graph and let $v \in V(G)$. Let $f_t(G, v)$ denote the smallest integer $m$ such that, whatever be the distribution of $m$ pebbles on the vertices of $G$, $t$ pebbles can be moved to $v$.

We define $t$-pebbling number $f_t(G)$ of a graph $G$ to be

$$f_t(G) = \max_{v \in V(G)} f_t(G, v)$$

**Sequential join [3]:** Let $G_1$ and $G_2$ be graphs such that $G_1$ and $G_2$ have disjoint vertex sets $V_1$ and $V_2$ and edge sets $E_1$ and $E_2$ respectively. Their join $G_1 + G_2$ consists of $G_1 \cup G_2$ and all edges joining $V_1$ with $V_2$. For three or more disjoint graphs $G_1, G_2, \ldots, G_n$, the sequential join $G_1 + G_2 + \cdots + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G3) \cup \cdots \cup (G_{n-1} + G_n)$.
Let

(i) $p(u)$ denotes the number of pebbles at the vertex $u$.

(ii) $p(W) = \sum_{v \in W} p(v)$

(iii) $p = |C| =$ the total number of pebbles used in $C$.

The following facts are almost folklore.

## 1. $t$-pebbling number of graphs

**Fact 1.1.**

$$f_t(K_1) = t$$

**Fact 1.2.**

$$f_t(K_2) = 2t$$

**Fact 1.3.** For any connected graph $G$, $f_t(G) \leq f_{t-1}(G) + f(G)$. Equality can hold in the above inequality. For example, if $G = P_n, n \geq 3$, $f_t(G) = t2^n = f_{t-1}(G) + f(G)$ for any $t \geq 2$.

More generally,

**Lemma 1.4.** Let $G$ be a connected graph of order $n \geq 2$. Let $u$ be any vertex in $G$. Then $f_t(G, u) \geq 2t + (n-2)$, if $n \geq 4$. Equality holds if $\deg(u) = n - 1$.

*Proof.* Let $u$ be the target vertex. Let $v \in N(u)$. Let $C$ be a pebbling configuration of $2t + (n-3)$ pebbles on the vertices of $G$ such that $p(u) = 0$, $p(v) = 2t - 1$, $p(w) = 1$, for all $w \in V(G) - \{u, v\}$. Then $t$ pebbles cannot be moved to $u$. This implies $f_t(G, u) \geq 2t + (n-2)$. Now, suppose that $\deg u = n - 1$, and $C$ is a configuration of $2t + n - 2$ pebbles on $V(G)$. Let $p(u) = k$. If $V_1 = V(G)\{u\}$, then there are $(2t + (n-2) - k) - (n-1) = 2t - 1$

extra pebbles on $V_1$. Hence $\left\lceil \frac{2t-1-k}{2} \right\rceil$ pebbles can be moved to $u$. Hence after a pebbling process, there will be $k + \left\lceil \frac{2t-1-k}{2} \right\rceil = t + \left(k - \left\lfloor \frac{k+1}{2} \right\rfloor\right) \geq t$ pebbles on $u$. Therefore, if $\deg u = n-1$, $f_t(G,u) = 2t + (n-2)$. Hence the lemma is proved. ∎

As a corollary we have the following.

**Corollary 1.5.** $f_t(K_n) = 2t + (n-2)$, if $n > 1$.

Hence, unless otherwise specified, hereafter we assume $G$ as a non-complete graph. In particular that $|V(G)| \geq 3$.

**Lemma 1.6.** Let $v$ be any vertex of degree less than $(n-1)$, in a connected graph $G$, of order $n$. Then $f_t(G,v) \geq 4t$.

*Proof.* Since $\deg(v)$ less than $(n-1)$, there exists a vertex $u$ of $G$ such that $u \notin N[v]$. Consider the configuration $C$ with $p = 4t-1$ such that $p(u) = 4t-1, p(x) = 0$, for every $x \neq u$. Then $t$ pebbles cannot be moved on to $v$ and hence $f_t(G,v) \geq 4t$. ∎

**Corollary 1.7.** For any vertex $v$ of degree less than $n-1$, we have $f_t(G,v) \geq \max\{4t, 2t+(n-2)\}$.

A class of graphs for which equality holds in the above inequality is considered in corollary (1.9).

We next consider the $t$-pebbling number for graphs of the form $G_1 + G_2$.

**Theorem 1.8.** Let $G$ be a graph with vertex set $V$ which can be partitioned into two nonempty subsets $V_1$ and $V_2$ which satisfy the following properties.

(i) $V_1 = \{u_1, u_2, \ldots, u_{r_1}\}$ is an independent set .

(ii) $V_2 = \{v_1, v_2, \ldots, v_{r_2}\}$ where $r_1$ and $r_2$ are integers $\geq 2$.

(iii) $N[u_1] = N[u_2] = \cdots = N[u_{r_1}] = V_2$. Then $f_t(G,u_1) = \max\{2t+n-2, 4t+r_1-2\}$.

*Proof.* We have seen already that $f_t(G,u) \geq 2t + (n-2)$ for any vertex $u \in V(G)$, for a graph $G$. Again consider the configuration $C$, given by $p(u_1) = p(V_2) = 0, p(u_2) = 4t-1, p(u_i) = 1$, for all $i, 3 \leq i \leq r_1$. Clearly we cannot move $t$ pebbles to $u_1$ starting from $p$. Hence $f_t(G,u_1) \geq \max(2t+n-2, 4t+r_1-2)$. Next, we prove the reverse inequality.

Let $V_3 = V_1 - \{u_1\}$. We prove the theorem by induction on $t$. When $t = 1$, the theorem reads as $f(G,u_1) = \max\{n, r_1+2\} = n$, since $r_1, r_2 \geq 2$ and $r_1 + r_2 = n$. Let $C$ be a configuration of $n$ pebbles on the vertices of $G$. If $p(u_1) > 0$ or $p(v_i) \geq 2$, for any vertex $v_i \in V_2, u_1$ can be pebbled. So, we may assume $p(u_1) = 0$ and $p(v_i) \leq 1$,for every $vi \in V_2 \Rightarrow p(V_3) \geq n - r_2 \geq r_1$. As $|V_3| = r_1 - 1$, there exists at least one $u_i \in V_3$ such that $p(u_i) \geq 2$. If $p(v_i) \geq 1$ for atleast one $i$, we can move a pebble at $u_1$ using the path $u_2 v_i u_1$. Otherwise, $p(V_2) = 0 \Rightarrow p(V_3) = n$. Since $n(r_1) = r_2 + 1 \geq 3$, atleast 3 excess pebbles are in $V_3$. Hence, two pebbles can be moved to $V_2$ and then one pebble can be moved to $u_1$. Hence $f(G, u_1) = n$. Therefore the result is true for $t = 1$. We may assume that $2 \Leftrightarrow n - r_1 t \Leftrightarrow 2t \leq r_2$. We may assume $t \geq 2$. We note that $2t + n - 2 \geq 4t + r_1 - \geq 2$.

**Case 1(a) :** Let $2t \leq r_2$.

In this case, we prove $f_t(G, u_1) = 2t + n - 2$ by induction on $t$. We have proved the result for $t = 1$. Assume that the result is true for $2, 3, \ldots, (t-1)$. Let $p$ be a distribution of $s = 2t + (n-2)$ pebbles on $G$. If either $p(u_1) > 0$ or $p(v_i) \geq 2$, for any vertex $v_i \in V_2, u_1$ can be pebbled at the cost of atmost two pebbles. Then by induction we can move $(t-1)$ more pebbles to $u_1$. Therefore, we may assume $p(u_1) = 0$ and $p(x) \leq 1, \forall x \in V_2$. Let $k = |\{x \mid x \in V_2, p(x) = 1\}| \Rightarrow p(V_3) = s - k$.

**Case (1) :** $k \geq t$.

In this case, $(s - k) = 2t + [(n-2) - k] \geq 2t$. We can move one pebble each to the $t$ occupied vertices in $V_2$. Then we can move $t$ pebbles to $u_1$.

**Case (2) :** $k < t$.

In this case, if we can move $k + 2(t-k) = 2t - k$ pebbles to $V_2$, we are through. For that, we can move the pebbles in such a way that each of the $k$ occupied vertices has an even number of pebbles, the total number being $2t$. From there $t$ pebbles can be moved to $u_1$. But $s - k = 2t + (n-2) - k \geq 4t - k \geq 4t - 2k$. Thus, it is possible to move $t$ pebbles to $u_1$ starting from $p$.

**Case 2(b) :** $2t = r_2 + 1$.

Let $C$ be a configuration of $s = 4t + r_1 - 2$ pebbles on $G$. We note that $p = 4t + r_1 - 2 = 2t + r_1 + r_2 + 1 - 2 = 2t + n - 1$. Also, as $2(t-1) < r_2$, by case 1, $f_{t-1}(G, u_1) = 2(t-1) + n - 2 =$ ï£¬$p$ï£¬ $- 3$. Consider the following cases,

(i) $p(u_1) \geq 1$

(ii) $p(ui) \geq 2$ for some $u_i \in V_3$ and $p(v_i) \geq 1$ for some $v_i \in V_2$.

(iii) $p(v_i) \geq 2$ for some $v_i \in V_2$.

In each of the above cases we can move one pebble to $u_1$ at the cost of atmost 3 pebbles, then we can move $(t-1)$ pebbles to $u_1$. We may therefore assume that none of the preceding conditions holds.

This implies $p(u_1) = 0, p(V_2) \leq r_2$ and $p(V_3) \geq 4t + r_1 - 2 - r_2 \geq 2t + (r_1 - 1)$. If now $p(v_i) = 1$ for some $v_i \in V_2$, we can move one pebble to $u_1$ at the cost of three pebbles. Then we can move $(t-1)$ more pebbles to $u_1$. So, we may assume further that $p(V_2) = 0$. Then $p(V_3) = 4t + r_1 - 2$. There are atleast $(4t + r_1 - 2) - (r_1 - 1) = 4t - 1$ extra pebbles on the vertices in $V_3$. Hence $t$ pebbles can be moved to $u_1$ from $V_3$ through a vertex in $V_2$.

**Case (c) :** We prove in general that, if $t \geq \left\lceil \frac{r_2}{2} \right\rceil$, $f_t(G, u_1) = 4t + r_1 - 2$. We prove the result by induction on $t$. We have already proved the result when $t = \left\lceil \frac{r_2}{2} \right\rceil$. So, we assume that $f_r(G, u_1) = 4r + r_2 - 2$ if $\left\lceil \frac{r_2}{2} \right\rceil \leq r \leq t - 1$. If $2t = r_2 + 1$ we have proved. When $2t > r_2 + 1, 2(t - 1) > r_2$. Whatever be the distribution of $4t + r_1 - 2$ pebbles on the vertices of $G$, one pebble can be moved to $u_1$ at the cost of atmost 4 pebbles. There are atleast $4(t - 1) + r_1 - 2$ pebbles remaining with $2(t - 1) \geq r_2$. Therefore the result is true by induction on $t$. The theorem is proved. ■

As mentioned earlier, we consider a class of graphs for which the equality holds in the inequality obtained in corollary (1.7).

**Corollary 1.9.** Let $G$ be a connected graph of order $n \geq 4$. Let $u$ and $v$ be non adjacent vertices of $G$ such that $\deg u = \deg v = n - 2$, then $f_t(G, v) = \max\{4t, 2t + (n - 2)\}$.

Proof. If $G_1 = \langle u \rangle, G_1 = G - \{u, v\}$ and $G_3 = \langle v \rangle$ , we see that $G$ is isomorphic to the sequential join $G_1 + G_2 + G3$. ■

**Corollary 1.10.** Let $K_{m_1, m_2, \ldots, m_r}$ be a complete $r$-partite graph. Let $n = \sum m_i, \forall i, 1 \leq i \leq r$. Then $f_t(G) = \max\{(2t + n - 2, \max(4t + mi - 2))\}$, where $1 \leq i \leq r$.

Next we consider the other extreme $G = G_1 + G_2$, where $G_1$ is complete.

**Theorem 1.11.** Let $G \cong K_{n_1} + K_{n_2} + K_{n_3}$, then

$$f_t(G) = \begin{cases} 2t + n - 2, & \text{if } n_2 \geq 2t \\ 2t + (n - 2) + (2t - n_2), & \text{if } n_2 \leq 2t \end{cases}.$$

*Proof.* Proof of theorem (1.11) is similar to the proof of theorem (1.8). ■

**Question 1.12.** Find the $t$-pebbling number of the sequential join of $r$ complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_r}$, for $r \geq 4$.

## 2. Bound for the $t$-pebbling number of graphs

Now, we establish some bounds for the $t$-pebbling number of graphs.

**Theorem 2.1.** If $G$ is a connected graph of order $n$. Let $d =$ diameter of $G$. Then $f_t(G) \geq \max\{t2^d, 2t + n - 2\}$.

*Proof.* Let $u_0, u_1, \ldots, u_d$ be a diametral path in $G$. Consider the pebbling configuration $C$, given by $p(u_d) = (t2^d) - 1, p(v) = 0$, for every $v \in V(G) - \{u_d\}$. Then it is not possible to move $t$ pebbles to $v_0 \Rightarrow ft(G) \geq t2^d$. Again, by lemma 5.3, $f_t(G) \geq 2t + n - 2. \Rightarrow ft(G) \geq \max\{t2^d, 2t + n - 2\}$. The theorem is proved. ■

Chan and Godbole [1] obtained an upper bound for the pebbling number of diameter d graph as $f(G) \leq (n-d)(2^d - 1) + 1$. Using exactly the same argument, we establish an upper bound for the $t$ pebbling number of diameter $d$ graphs.

**Theorem 2.2.** Let $G$ be a connected graph of order $n$. Let $d = \text{diam}(G)$. Then $f_t(G) \leq (t-1)2^d + (n-d)(2^d - 1) + 1$.

*Proof.* Proof is by induction on $t$. When $t = 1$, the inequality reads as $f(G) \leq (n - d)(2^d - 1) + 1$, which is true (from [1]). When $t \geq 2$, let $u \in V(G)$. Let $e$ be the eccentricity of $u$. Let $C$ be a configuration of $(n-e)((t2e) - 1) + 1$ pebbles on $V(G)$. Let $u_0 = u, u_1, u2, \ldots, u_e$, be an eccentric path with $d(u_0, u_e) = e$. [An eccentric path of a vertex $v$ in $G$ is a path of length $e(v)$ starting from $v$ in $G$ and ending at $u$ where $d(u, v) = e(v)$].

Consider the sets $S_0 = \{u, u_1, u2, \ldots, u_e\}, S_i = \{x_i\}, 1 \leq i \leq n - e - 1$ where the remaining vertices in $G = V(G) - S_0 = \{x_1, x_2, \ldots, x_{n-e-1}\}$. Thus, in $p$, at least one set must receive $(t2^e)$ pebbles and $t$ pebbles can be moved to $u$. Thus $f_t(G, u) \leq (n-e)((t2^e) - 1) + 1$.

We note that $(n-e-1)(t2^{e+1} - 1)(n-e)((t2^e) - 1) = (n-e)(t2^e)(t2^e - 1) \geq 1$, if $e < d \leq n - 1$. Thus, $(n-e)((t2^e) - 1) + 1 \leq (n-d)((t2^d) - 1) + 1$. ■

**Remark 2.3.** We note that equality is attained in the inequality for paths.

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