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Induced Δ-Decomposition of Graphs

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Abstract

A decomposition of a graph G is a collection of edge-disjoint subgraphs $H_1, H_2, \dots H_r$ of G such that every edge of G belongs to exactly one H_i. A decomposition $\psi = \{H_1, H_2, \dots\dots, H_r\}$ is called a Δ-*decomposition* if the maximum degree of H_i is i for each i. A Δ-decomposition $\psi = \{H_1, H_2, \dots\dots, H_r\}$ of a graph G is called an *induced* Δ-*decomposition* if each of the subgraph H_i is an induced subgraph of G. The minimum cardinality of the induced Δ-decomposition of a graph G is called the *induced* Δ-*decomposition number* of G and is denoted by $\pi_{i\Delta}(G)$. Δ-decomposition concept was initiated in [1]. In this paper, we continue this study by defining induced Δ-decomposition and the corresponding parameter $\pi_{i\Delta}$. We determine the value of $\pi_{i\Delta}$ for perfect facto graph and some standard graphs.

Keywords: Induced Δ-decomposition, induced Δ-decomposition number.

Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected graph without loops or multiple edges. In a graph G, p and q denote respectively the order and size of G. Also $\Delta(G)$ denotes the maximum degree of G and is given by $\Delta(G) = max\{d(v)/v \in V\}$. For terms not defined here we refer to Harary [5].

A *decomposition* of a graph G is a collection of edge-disjoint subgraphs $H_1, H_2, \dots H_r$ of G such that every edge of G belongs to exactly one H_i. A decomposition $\psi = \{H_1, H_2, \dots\dots, H_r\}$ is called a Δ-*decomposition* if $\Delta(H_i) = i$ for each i. The minimum cardinality of the Δ-decomposition of a graph G is called the Δ-decomposition number of G and is denoted by $\pi_{i\Delta}(G)$. Various types of decompositions have been studied by various authors. Induced Path decomposition number, Induced Acyclic Path decomposition number are some such decomposition parameters.

1. For any positive integer z, we define the *z-facto graph* as a graph $G(z) = (V, E)$ where $V = \{v_i \,/\, v_i \; is\ a\ factor\ of\ z\}$ and two vertices v_i and v_j are adjacent if and only if $v_i v_j \in V$. A graph G is said to be a *facto graph* if there exists a positive integer z such that G is isomorphic to an z-facto graph $G(z)$, for some z. If $G \cong G(z)$, for some z then the *integral order of the graph G* is equal to z and is denoted by $o_i(G)$. We refer a facto graph $G(z)$ by G with $o_i(G) = z$.
2. A facto graph G which is of integral order p^{α}, where p is a prime and α is a positive integer is called a *Perfect Facto graph*. If α is odd (even) then G is called an *odd* (*even*) *perfect facto graph*.

Example 3.5: Odd perfect facto graphs G of integral order p^9 is depicted in Figure 1.

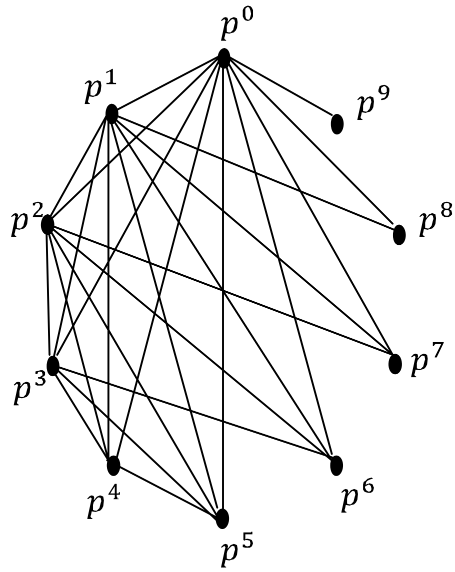


Figure 1

Induced Δ-decomposition of a graph

In this section we introduce the notion of induced Δ-decomposition and induced Δ-decomposition number and determine the value of this parameter for even perfect facto graph and its complement.

Definition 1.1: [3] A decomposition $\psi = \{H_1, H_2, \ldots\ldots, H_r\}$ of a graph G is called a Δ-decomposition if $\Delta(H_i) = i$, for $i = 1,2, \ldots\ldots, r$. The minimum cardinality of a Δ-

decomposition of G is called the Δ - decomposition number of G and is denoted by $\pi_\Delta(G)$.

Definition 1.2: A Δ - decomposition $\psi = \{H_1, H_2, \dots\dots, H_r\}$ of a graph G is called an induced Δ - decomposition of G if each H_i is an induced subgraph of G. The minimum cardinality of the induced Δ - decomposition of G is called the induced Δ - decomposition number of G and is denoted by $\pi_{i\Delta}(G)$. An induced Δ- decomposition ψ of G with $|\psi| = \pi_{i\Delta}(G)$ is called the minimum induced Δ - decomposition of G.

Example 1.3: Consider the graph G given in Figure 1.

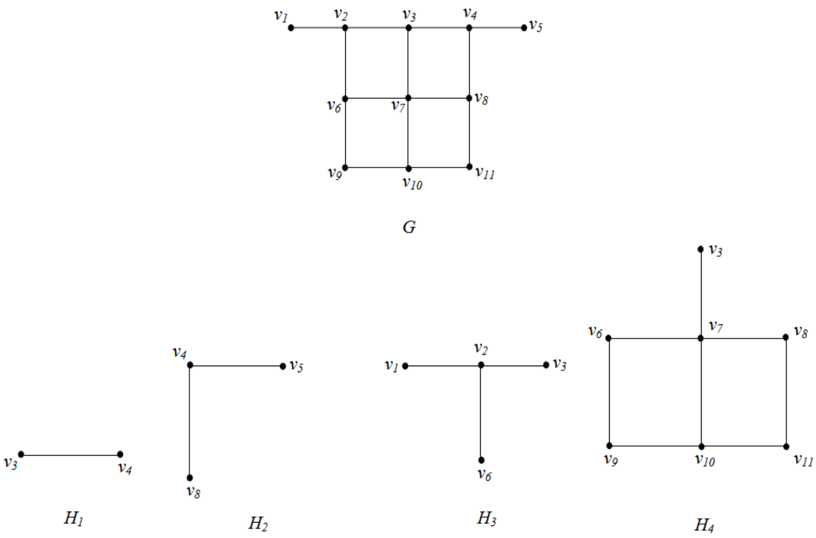


Figure 1

Let $\psi = \{H_1, H_2, H_3, H_4\}$. The subgraph H_1 of G is induced by $\{v_3, v_4\}$, H_2 is induced by $\{v_4, v_5, v_8\}$, H_3 is induced by $\{v_3, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$. Also $\Delta(H_i) = i$, for $i = 1, 2, 3, 4$. Hence ψ is an induced Δ - decomposition of G.

Now consider another decomposition of G which is depicted in Figure 2.

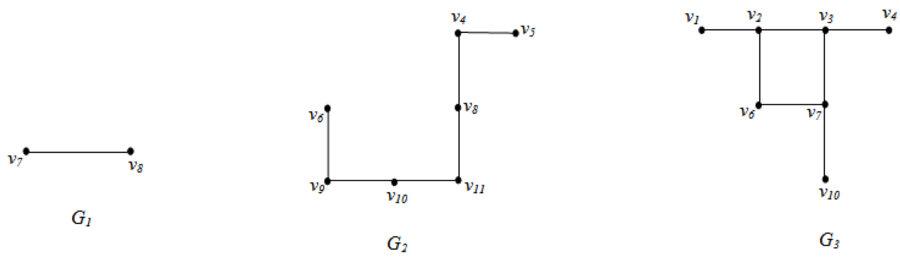


Figure 2

Here G_1 is induced by $\{v_7, v_8\}$, G_2 is induced by $\{v_6, v_9, v_{10}, v_{11}, v_8, v_4, v_5\}$ and G_3 is induced by $\{v_1, v_2, v_3, v_4, v_6, v_7, v_{10}\}$. Let $\psi_1 = \{G_1, G_2, G_3\}$. Then ψ_1 is an induced Δ - decomposition of G. Also, $\pi_{i\Delta}(G) = |\psi_1| = 3$.

Induced Δ-decomposition of perfect facto graph

Theorem 1.4: Every odd perfect facto graph G with $o_i = p^\alpha$, has an induced Δ - decomposition and $\pi_{i\Delta}(G) = \left\lceil \frac{\alpha}{2} \right\rceil$.

Proof: Let $G = (V, E)$ be an odd perfect facto graph and $o_i(G) = p^\alpha, \alpha \in Z^+$ and α is odd. Let $V = \{p^0, p^1, p^2, \dots \dots, p^\alpha\}$ and we have the edge set

$$E = \left\{p^0 p^i / 1 \leq i \leq \alpha\right\} \cup \left\{p^1 p^i / 2 \leq i \leq \alpha - 1\right\} \cup \dots \dots \dots \cup \left\{p^{\left\lfloor \frac{\alpha}{2} \right\rfloor} p^{\left\lceil \frac{\alpha}{2} \right\rceil}\right\}.$$

We have the maximum clique of G is the complete subgraph $K_{\left\lceil \frac{\alpha}{2} \right\rceil + 1}$ of G and the clique number is $\left\lceil \frac{\alpha}{2} \right\rceil + 1$. Also $K_{\left\lceil \frac{\alpha}{2} \right\rceil + 1}$ is induced by the set $\{p^0, p^1, p^2, \dots \dots, p^{\left\lceil \frac{\alpha}{2} \right\rceil}\}$ and it follows that $\Delta\left(K_{\left\lceil \frac{\alpha}{2} \right\rceil + 1}\right) = \left\lceil \frac{\alpha}{2} \right\rceil$.

Now for $i = 0,1,2, \dots \dots, \left\lfloor \frac{\alpha}{2} \right\rfloor - 1$, let $E\left(K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor - i}\right) = \left\{p^i p^j / \left\lceil \frac{\alpha}{2} \right\rceil + 1 \leq j \leq \alpha - i\right\}$.

When $i = 0$, we have $E\left(K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor}\right) = \left\{p^0 p^{\left\lceil \frac{\alpha}{2} \right\rceil + 1}, p^0 p^{\left\lceil \frac{\alpha}{2} \right\rceil + 2}, \dots \dots, p^0 p^\alpha\right\}$ and hence $\Delta\left(K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor}\right) = \left\lfloor \frac{\alpha}{2} \right\rfloor$.

When $i = 1$, we have $E\left(K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor - 1}\right) = \left\{p^1 p^{\left\lceil \frac{\alpha}{2} \right\rceil + 1}, p^1 p^{\left\lceil \frac{\alpha}{2} \right\rceil + 2}, \dots \dots, p^1 p^{\alpha - 1}\right\}$ and $\Delta\left(K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor - 1}\right) = \left\lfloor \frac{\alpha}{2} \right\rfloor - 1$.

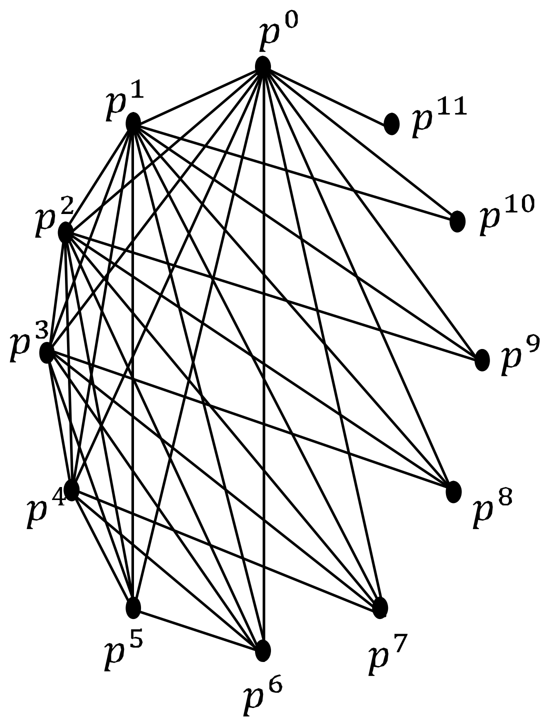
On incrementing the value of i by 1 repeatedly we get stars $K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor - 2}, K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor - 3}, \dots$and after $\left\lfloor \frac{\alpha}{2} \right\rfloor - 1$ steps we have $i = \left\lfloor \frac{\alpha}{2} \right\rfloor - 1$and $E(K_{1,1}) = \left\{p^{\left\lfloor \frac{\alpha}{2} \right\rfloor - 1} p^j / \left\lceil \frac{\alpha}{2} \right\rceil + 1 \leq j \leq \left\lfloor \frac{\alpha}{2} \right\rfloor + 1\right\} = \left\{p^{\left\lfloor \frac{\alpha}{2} \right\rfloor - 1} p^{\left\lfloor \frac{\alpha}{2} \right\rfloor + 1}\right\}$ and $\Delta(K_{1,1}) = 1$.

Now let $\psi = \left\{K_{\left\lfloor \frac{\alpha}{2} \right\rfloor + 1}\right\} \cup \left\{\bigcup_{i=0}^{\left\lfloor \frac{\alpha}{2} \right\rfloor - 1} \left(K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor - i}\right)\right\}$.

The decomposition ψ is an induced Δ - decomposition of G, since each subgraphs are induced subgraphs of G and also $\Delta\left(K_{\left\lfloor \frac{\alpha}{2} \right\rfloor + 1}\right) = \left\lceil \frac{\alpha}{2} \right\rceil$ and for $i = 0,1, \dots, \left\lfloor \frac{\alpha}{2} \right\rfloor - 1$, $\Delta\left(K_{1, \left\lfloor \frac{\alpha}{2} \right\rfloor - i}\right) = \left\lfloor \frac{\alpha}{2} \right\rfloor - i$. Hence $\pi_{i\Delta}(G) \leq |\psi| = 1 + \left\lfloor \frac{\alpha}{2} \right\rfloor = \left\lceil \frac{\alpha}{2} \right\rceil$.

Also since $K_{\left\lceil \frac{\alpha}{2} \right\rceil + 1}$ is an induced subgraph of G which is a maximum clique of G we have $\pi_{i\Delta}(G) \geq \left\lceil \frac{\alpha}{2} \right\rceil$. Hence $\pi_{i\Delta}(G) = \left\lceil \frac{\alpha}{2} \right\rceil$.

Example 1.5: Consider the odd perfect facto graph G of integral order p^{11}, which is depicted in the following Figure.



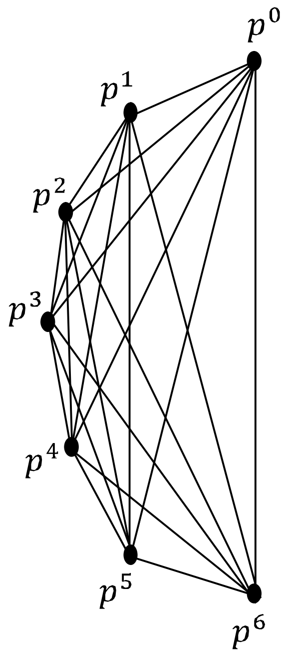
G

Figure

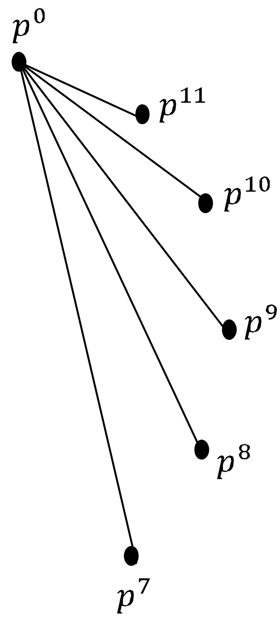
Here $\left\lceil \frac{\alpha}{2} \right\rceil = \left\lceil \frac{11}{2} \right\rceil = 6$ and $\left\lfloor \frac{\alpha}{2} \right\rfloor = 5$. The maximum clique of $G$and the other induced subgraph are constructed as below:

The maximum clique K_7 is induced by the set $\{p^0, p^1, p^2, \dots \dots, p^6\}$ and $K_{1,5}$ is induced by the set $\{p^0, p^7, p^8, p^9, p^{10}\}$ $K_{1,4}$ is induced by the set $\{p^1, p^7, p^8, p^9, p^{10}\}$. The subgraph $K_{1,3}$ is induced by the set $\{p^2, p^7, p^8, p^9\}$, $K_{1,2}$ is induced by $\{p^3, p^7, p^8\}$ and $K_{1,1}$ is induced by the set of vertices $\{p^4, p^7\}$.

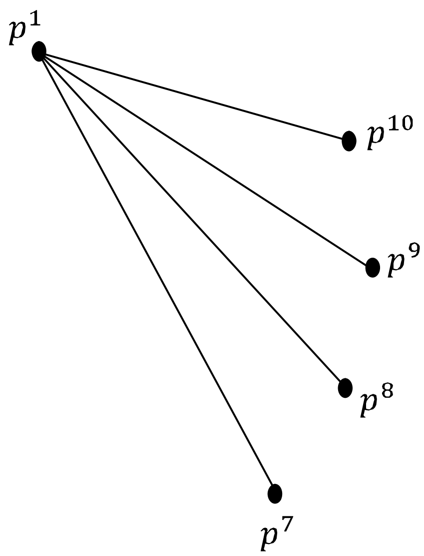
The induced Δ - decomposition of G is depicted in Figure 3.



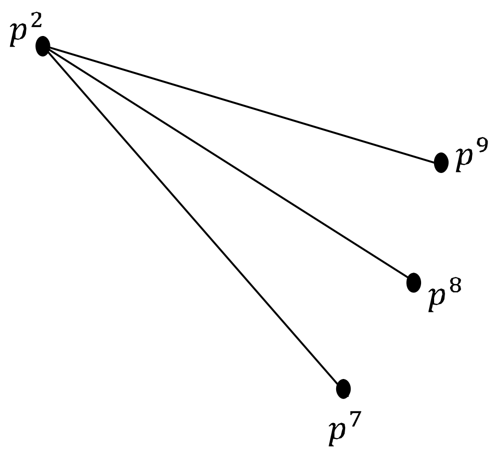
K_7



$K_{1,5}$



$K_{1,4}$



$K_{1,3}$

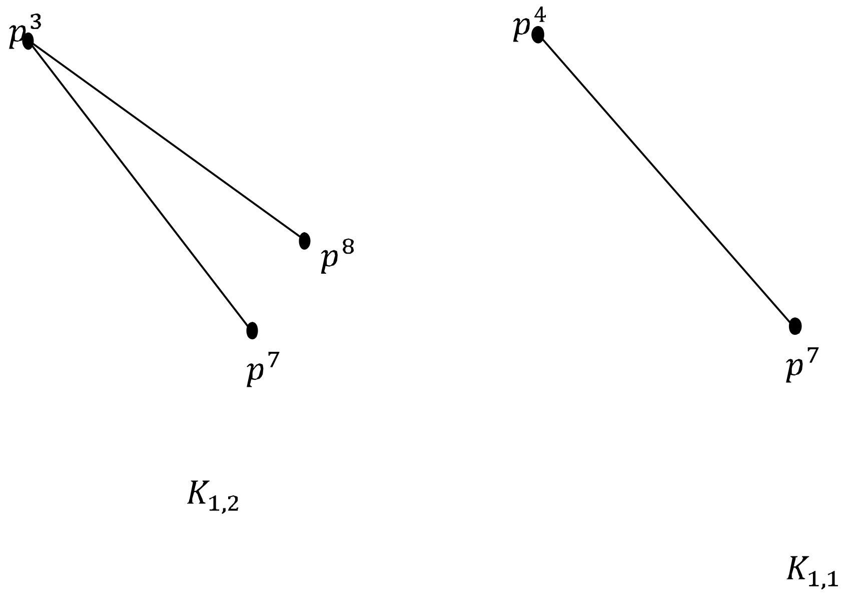


Figure 3

From the figure, we observe that the decomposition $\psi = \{K_7, K_{1,5}, K_{1,4}, K_{1,3}, K_{1,2}, K_{1,1}\}$ is an induced Δ - decomposition of G and also $\pi_{i\Delta}(G) = 6$.

Theorem 1.6: If G is an even perfect facto graph with $o_i(G) = p^{\alpha}$, then $\pi_{i\Delta}(\bar{G}) = \frac{\alpha}{2}$.

Proof: G is an even perfect facto graph with $o_i(G) = p^{\alpha}$, α is even.

By Theorem 2.19, the complement $\bar{G}$ of G is isomorphic to $G' \cup K_1$, where $o_i(\bar{G}) = p^{\alpha-1}$.

G' is an odd perfect facto graph and hence by Theorem 2.1, $\pi_{i\Delta}(G') = \left\lceil\frac{\alpha-1}{2}\right\rceil$. Since $\alpha - 1$ is odd, $\left\lceil\frac{\alpha-1}{2}\right\rceil = \frac{\alpha-1+1}{2} = \frac{\alpha}{2}$. Thus $\pi_{i\Delta}(\bar{G}) = \frac{\alpha}{2}$.

Now $\bar{G} \cong G' \cup K_1$ implies that $\pi_{i\Delta}(\bar{G}) = \pi_{i\Delta}(G' \cup K_1)$. Also, since K_1 has no edges, we have $\pi_{i\Delta}(\bar{G}) = \pi_{i\Delta}(G') = \frac{\alpha}{2}$.

Remark 1.7: The even perfect facto graph has no induced Δ - decomposition.

Induced Δ - decomposition and of Graphs

In this section we determine the induced Δ-decomposition number for the path and for the particular cases of star for which the induced Δ-decomposition exist.

Theorem 2.1: $\pi_{i\Delta}(K_{1,n}) = \begin{cases} 2 \text{ if } n = 3 \\ m \text{ iff } n = \binom{m}{2} \text{ and } m > 2 \end{cases}$

Proof: Let $V = \{u, v_1, v_2, \dots, v_n\}$ be the vertex set of the star $K_{1,n}$ with centre u. When $n = 3$, then the star $K_{1,3}$ can have induced Δ-decomposition, $\psi = \{K_{1,1}, K_{1,2}\}$ where $K_{1,1}$ is induced by the set $\{u, v_1\}$ and $K_{1,2}$ is induced by $\{u, v_2, v_3\}$ clearly ψ is the minimum induced Δ-decomposition of $K_{1,3}$ and hence $\pi_{i\Delta}(K_{1,3}) = 2$.

Now, suppose that the star $K_{1,n}$ has an induced Δ - decomposition and let $m > 2$ and $\psi = \{H_1, H_2, \dots, H_m\}$ be an induced Δ-decomposition of $K_{1,n}$.Then $\Delta(H_i) = i$, for $i = 1,2, \dots, m$. Since any induced subgraph of $K_{1,n}$ having maximum degrees $1,2,3, \dots, m, H_i$ must be isomorphic to $K_{1,i}$. Also if q_i is the size of H_i, then q_i must be equal to i.

Then the size q of $K_{1,n}$ which is equal to n is given by

$$q = n = 1 + 2 + 3 + \cdots + m = \binom{m}{2}$$

Conversely suppose that $n = \binom{m}{2}, m > 2$.

The star $K_{1,n}$ is now $K_{1,\binom{m}{2}}$, and it has $\binom{m}{2} = 1 + 2 + 3 + \cdots + m$ edges. The star $K_{1,\binom{m}{2}}$ can be decomposed into induced subgraphs $K_{1,1}, K_{1,2}, K_{1,3}, \dots, K_{1,m}$, induced by the set $\{u, v_1\}, \{u, v_2, v_3\}, \{u, v_4, v_5, v_6\}, \dots, \left\{u, u_{\binom{m-1}{2}+1}, u, u_{\binom{m-2}{2}+2}, \dots, u_{\binom{m}{2}}\right\}$ respectively and $\Delta(K_{1,i}) = i, i = 1,2, \dots, m$.Hence the decomposition,$\psi = \{K_{1,1}, K_{1,2}, \dots, K_{1,m}\}$ is an induced Δ - decomposition and this is the only possible number of Δ - decomposition, so that $\pi_{i\Delta}(K_{1,n}) = m$.

Result: 2.2 For any path P_n with $n > 3, \pi_{i\Delta}(P_n) = 2$.

Proof: Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of P_n and let the path be denoted by $v_1, v_2, \dots, v_n$. Consider the complete graph K_2 induced by $\{v_1, v_2\}$ and let P_{n-1} be the path given by $v_2, v_3, v_4, \dots, v_n$ and is induced by $\{v_2, v_3, \dots, v_n\}$.

Now, $\Delta(K_2) = 1$ and since $n > 3, \Delta(P_{n-1}) = 2$.

Thus the decomposition $\psi = \{K_2, P_{n-1}\}$ is an induced Δ - decomposition of P_n. Also, P_n cannot have other induced Δ - decompositionlity different from 2 and hence $\pi_{i\Delta}(P_n) = 2$.

Result 2.3: A cycle C_n has no induced Δ - decomposition.
If $C_n, n > 3$ be given by $v_1, v_2, \dots, v_n, v_1$ and if K_2 is induced by $\{v_1, v_2\}$, then all the remaining edges cannot be covered by a path. More precisely, the edges incident with v_1 and v_2 cannot be covered by an induced path, and hence C_n has no induced Δ - decomposition.

Remark 2.4: The complete graph has no induced Δ - decomposition.

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