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Characterization Of Various G-Inverses Of Intuitionistic Fuzzy Matrices

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Abstract:

In this paper, we represent an intuitionistic fuzzy matrix as the Cartesian product representation of its membership and non-membership matrices. By using this representation, we shall discuss the characterization of set of all g-inverses of an IFM and characterized the set of various g-inverses associated with the IFM.

Keywords: Fuzzy matrix, Intuitionistic fuzzy matrix, g-inverse.

1. Introduction:

We deal with fuzzy matrices that is, matrices over the fuzzy algebra F^M and F^N with support [0,1] and fuzzy operations {+, .} defined as a + b = max {a, b}, a.b = min{a,b} for all $a,b \in F^M$ and a + b = min {a, b}, a .b = max{a, b} for all $a, b \in F^N$. Let F^M_{mxn} be the set of all mxn Fuzzy matrices over F. A matrix $A \in F^M_{mxn}$ is said to be regular if there exists $X \in F^M_{nxm}$ such that AXA = A, X is called a generalized inverse (g-inverse) of A. In [3], Kim and Roush have developed the theory of fuzzy matrices, under max min composition analogous to that of Boolean matrices. Cho [2] has discussed the consistency of fuzzy matrix equations, if A is regular with a g-inverse X, then b.X is a solution of x A=b. Further every invertible matrix is regular. For more details on fuzzy matrices one may refer [4]. Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. The concept of intuitionistic fuzzy matrices (IFMs) as a generalization of fuzzy matrix was studied and developed by Madhumangal Pal et.al.[6]. In [7], Sriram and Murugadas have derived the equivalent condition for the existence of the generalized inverses. In our earlier work, we have studied on regularity of IFM [5].

In this paper, we discussed the characterization of set of all various g-inverses of an IFM.

2. Preliminaries

Let $(IF)_{mxn}$ be the set of all intuitionistic fuzzy matrices of order mxn. Let $(IF)_{mxn}$ be the set of all intuitionistic fuzzy matrices of order mxn. First we shall represent A $\in (IF)_{mxn}$ as Cartesian product of fuzzy matrices. The Cartesian product of any two matrices A=$(a_{ij})_{mxn}$ and B=$(b_{ij})_{mxn}$, denoted as $\langle A,B\rangle$ is defined as the matrix whose ijth entry is the ordered pair $\langle A,B\rangle = (\langle a_{ij}, b_{ij}\rangle)$. For $A = (a_{ij})_{mxn}$ = $(\langle a_{ij\mu}, a_{ij\nu}\rangle)$ $\in (IF)_{mxn}$. We define $A_\mu = (a_{ij\mu}) \in$ F^M_{mxn} as the membership part of A and $A_\nu = (a_{ij\nu}) \in$ F^N_{mxn} as the non membership part of A. Thus A is the Cartesian product of A_μ and A_ν written as $A = \langle A_\mu, A_\nu\rangle$ with $A_\mu \in F^M_{mxn}$, $A_\nu \in F^N_{mxn}$.

Here we shall follow the matrix operations on intuitionistic fuzzy matrices as defined in our earlier work [5].

For $A, B \in (IF)_{mxn}$, if $A = \langle A_\mu, A_\nu\rangle$ and $B = \langle B_\mu, B_\nu\rangle$, then

(2.1) $$A + B = \langle A_\mu + B_\mu, A_\nu + B_\nu\rangle$$

For $A \in (IF)_{mxp}$, $B \in (IF)_{pxn}$ if $A = \langle A_\mu, A_\nu\rangle$ and $B = \langle B_\mu, B_\nu\rangle$, then

(2.2) $$AB = \langle A_\mu . B_\mu,\ A_\nu . B_\nu\rangle$$

$A_\mu . B_\mu$ is the max min product in F^M_{mxn},

$A_\nu . B_\nu$ is the min max product in F^N_{mxn}.

For $A \in (IF)_{mxn}$, R(A) (C(A)) be the space generated by the rows (columns) of A. Let us define the order relation on $(IF)_{mxn}$ as ,

(2.3) $$A \le B \Leftrightarrow A_\mu \le B_\mu \text{ and } A_\nu \ge B_\nu \Leftrightarrow A + B = B .$$

Definition 2.1[5]:
An $A \in (IF)_{mxn}$ is said to be regular if there exists $X \in (IF)_{nxm}$ satisfying AXA = A and X is called a generalized inverse (g-inverse) of A. which is denoted by A^-. Let A{1} be the set of all g-inverses of A.

Definition 2.2:
For an IFM A of order m x n, an IFM X of order n x m is said to be {1, 2}-inverse or semi inverse of A , if AXA = A and XAX = X

X is said to be {1,3}-inverse or a least square g-inverse of A ,if AXA = A and $(AX)^T$ = AX.

X is said to be {1,4}-inverse or a minimum norm g-inverse of A ,if AXA = A and $(XA)^T$ = XA .

X is said to be a Moore-Penrose inverse of A, if AXA = A, XAX = X, $(AX)^T$ = AX and $(XA)^T$ = XA .The Moore-Penrose inverse of A is denoted by A^+.

Lemma 2.3[5]:

Let $A \in (IF)_{mxn}$ be of the form $A = \langle A_\mu, A_\nu \rangle$. Then A is regular $\Leftrightarrow$ A_μ is regular in F^M_{mxn} under max min composition and A_ν is regular in F^N_{mxn} under min max composition.

Lemma 2.4[5]:

If $A \in (IF)_{mxn}$ is of the form $A = \langle A_\mu, A_\nu \rangle$, then (i) R(A) = $\langle R(A_\mu), R(A_\nu) \rangle$ and (ii) C (A) = $\langle C(A_\mu), C(A_\nu) \rangle$.

3. Characterization of various g-inverse:

In this section, we derive the characterization of the set of A{1} in terms of a particular element of the set.

Let A,B $\in (IF)_{mxn.}$ If A≥ B then by (2.3) $A_\mu \geq B_\mu$ and $A_\nu \leq B_\nu$. Let A_μ-B_μ =H_μ and A_ν+B_ν=H_ν are an IFMs, but $A_\mu \neq B_\mu + H_\mu$ and $A_\nu \neq H_\nu - B_\nu$. Therefore A ≥ B then A-B = H is an IFM, but A≠B+H.

Lemma 3.1:

For $A \in (IF)_{mxn}$ if G* and G are g-inverse of A such that G* ≥ G, then G+H is a g-inverse of A for some $H \in (IF)_{nxm}$ such that G* ≥ G + H ≥ G.

Proof:

Let G* - G = H. Then G* ≥ H.Since G* ≥ G and G* ≥ H, it follows that

G* ≥ G+H ≥ G.Then AG* A ≥ A(G+H) A ≥ AGA

⇒ A ≥ A(G+H)A ≥ A

⇒ A (G+H) A = A

Thus (G+H) is a g-inverse of A.

Theorem 3.2:

Let $A \in (IF)_{mxn}$ and G be a particular g-inverse of A. Then

$A_G\{1\}$ = {G+H/ for all $H \in (IF)_{nxm}$ such that A ≥ AHA} (3.1)

is the set of all g-inverse of A dominating G.

Proof:

Let B denote the set on the R.H.S of (3.1). Suppose $G^* \in A_G\{1\}$, then G* ≥ G.

Let G* - G = H. By Lemma (3.1), G* ≥ G+H ≥ G and G+H is a g-inverse of A dominating G. Further A(G+H) A = A ⇒ AGA + AHA = H

⇒ A+AHA = A

$\Rightarrow A \geq AHA.$

Hence G +H $\in$ B. Thus for each $G^* \in A_G\{1\}$ there exist a unique element in B.

Conversely, for any $G^* \in B$, $G^* = G+H \geq G$, with $A \geq AHA$

Now, $AG^*A = A(G+H)A = AGA + AHA = A + AHA = A$.Thus $G^* \in A_G\{1\}$.Hence the proof.

Corollary 3.3:

Let $A \in (IF)_n$ be an idempotent IFM.Then$\{G+H/\text{for all } H \in (IF)_n \text{ such that } A \geq AHA\}$…(3.2) is the set of all g-inverses of A dominating A.

Proof:

This follows from Theorem (3.2) by taking G = A. Since A is an idempotent IFM, A itself is a g-inverse.

Next we discuss the characterization of the sets A{1,3} and A{1,4} in terms of a particular element of the set.

Theorem 3.4:

The set A{1,3} consists of all solutions for X of AX =AG. Where G is a {1,3} inverse of A.

Proof:

Since $G \in A\{1,3\}$, by Definition (2.2), AGA = A and $(AG)^T = AG$. For $X \in A\{1,3\}$ we have AXA = A and $(AX)^T = AX$. Then

$$\begin{aligned} AG &= (AXA)G = (AX)(AG) \\ &= (AX)^T (AG)^T = (X^TA^T)(G^TA^T) \\ &= X^T (A^TG^TA^T) = X^TA^T \\ &= (AX)^T \quad \text{(By Definition (2.1))} \\ &= AX \end{aligned}$$

Hence X is a solution of AX = AG.

Conversely, let AG = AX with $G \in A\{1,3\}$.Then A = AGA

$\Rightarrow A = AXA$

$\Rightarrow X \in A\{1\}$ (3.3)

Since $AG = AX \Rightarrow (AG)^T = (AX)^T$

$$\begin{aligned} &\Rightarrow AG = (AX)^T \\ &\Rightarrow AX = (AX)^T \\ &\Rightarrow X \in A\{3\} \qquad (3.4) \end{aligned}$$

From (3.3) and (3.4) ,it follows that $X \in A\{1,3\}$.Hence the proof.

Theorem 3.5:

For $A \in (IF)_{mxn}$ and $G \in A\{1,3\}$,$A_G\{1,3\} = \{G+H/ \text{ for all } H \in (IF)_{mxn} \text{ such that } AG \geq AH\}$ …(3.5) is the set of all {1,3} inverses of A dominating G.

Proof:

Let B denote the set on the R.H.S of (3.5). Suppose $G^* \in A_G\{1,3\}$, then $G^* \geq G$. Let $G^* - G = H$. Since $A_G\{1,3\} \subseteq A_G\{1\}$, by theorem (3.2), $G^* \geq G+H \geq G$.

$\Rightarrow \quad AG^* = A(G+H) \geq AG$

By Theorem (3.4), $G^* \in A_G\{1,3\}$ and $G \in A_G\{1,3\}$

$\Rightarrow \quad AG^* = AG$

$\Rightarrow \quad A(G+H) = AG$

$\Rightarrow \quad AG \geq AH.$

Hence $G+H \in B$. Thus for each $G^* \in A_G\{1,3\}$, there exists an unique element in B.

Conversely for any $G^* \in B, G^* = G+H \geq G$ with $AG \geq AH$. Hence $AG^* = AG+AH = AG$. By Theorem (3.4), it follows that $G^* \in A_G\{1,3\}$. Hence the theorem.

Corollary 3.6:

For $A \in (IF)_n$ be a symmetric idempotent fuzzy matrix then {A+H/ for all $H \in (IF)_n$ such that AG/AH} is the set of all {1,3} inverses of A dominating A.

Proof:

This follows from Theorem (3.5) by taking G =A. Since A is symmetric and idempotent IFM, A itself is a {1,3} inverse.

Remark 3.7:

The condition that G is a {1,3} inverse of A is essential. This is illustrated in the following example.

Example 3.8:

$$For \quad A = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle, \ \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \in A\{1,3\}$$

$$\Rightarrow \quad A\{1,3\} \neq \Phi$$

$$\text{Consider} \quad G = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .7 & 0 \end{pmatrix} \right\rangle \notin A\{1,3\}$$

$$AG = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .7 & 0 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .7 & 0 \end{pmatrix} \right\rangle$$

$$\text{For} \quad H = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .3 & 0 \end{pmatrix} \right\rangle$$

$$AH = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .3 & 0 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .3 & 0 \end{pmatrix} \right\rangle$$

$\Rightarrow AG \geq AH$

but G+H $= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .7 & 0 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .3 & 0 \end{pmatrix} \right\rangle$

$$= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ .3 & 0 \end{pmatrix} \right\rangle$$

$\Rightarrow G+H \notin A\{3\}$

$\Rightarrow G+H \notin A_G\{1,3\}$

Since $G \in A\{1,3\} \Leftrightarrow G^T \in A^T\{1,4\}$

Theorem 3.9:

The set A{1,4} consists of all solutions for X of XA = GA, where G is a {1,4} inverse of A.

Proof:

This can be proved in the same manner as that of Theorem (3.4).

Theorem 3.10:

For $A \in (IF)_{mxn}$ and $G \in A\{1,4\}, A_G\{1,4\} = \{G+H/\text{for all } H \in (IF)_{nxm} \text{ such that } GA \geq HA\}$ …(3.6)

is the set of all {1,4} inverse of A dominating G.

Proof:

Let B denote set on the R.H.S of (3.6).Suppose $G^* \in A_G\{1,4\}$ then $G^* \geq G$. Let $G^* - G = H$. Since $A_G\{1,4\} \subseteq A_G\{1\}$, by lemma (3.1), $G^* \geq G+H \geq G$. This implies that $G^*A \geq (G+H)A = GA$

By Theorem (3.9), $G^* \in A_G\{1,4\}$ and $G \in A_G\{1,4\}$

$\Rightarrow G^*A = GA$

$\Rightarrow (G+H)A = GA$

$\Rightarrow GA \geq HA$

Thus $G+H \in B$. Hence for each $G^* \in A_G\{1,4\}$, there exists an unique element in B.

Conversely, for any $G^* \in B$, $G^* = G+H \geq G$ with $GA \geq HA$. Hence $G^*A = GA+HA = GA$. By Theorem (3.9), it follows that $G^* \in A_G\{1,4\}$. Hence the proof.

Corollary 3.11:

Let $A \in (IF)_n$ be a symmetric and idempotent intuitionistic fuzzy matrix. Then $\{A+H/\text{for all } H \in (IF)_n \text{ such that } GA \geq HA\}$ is the set of all {1,4} inverse of A dominating A.

Proof:
This follows from Theorem (3.10) by taking G=A. Since A is a symmetric idempotent IFM, A itself is a {1, 4} inverse.

Remark 3.12:
In Theorem (3.10), G is a {1,4} inverse of A is essential. This is illustrated in the following example.

Example 3.13:

For A = $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \in A\{1,4\}$

$\Rightarrow A\{1,4\} \neq \phi$

Consider G $= \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & .7 \\ 0 & 0 \end{pmatrix} \right\rangle$

G A $= \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & .7 \\ 0 & 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$

$= \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & .7 \\ 0 & 0 \end{pmatrix} \right\rangle$

For H $= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix} \begin{pmatrix} 0 & .3 \\ 0 & 0 \end{pmatrix} \right\rangle$

HA $= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix}, \begin{pmatrix} 0 & .3 \\ 0 & 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle$

$= \left\langle \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix}, \begin{pmatrix} 0 & .3 \\ 0 & 0 \end{pmatrix} \right\rangle$

$\Rightarrow GA \geq HA$

But G+H $= \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & .7 \\ 0 & 0 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} 1 & 0 \\ 0 & .2 \end{pmatrix}, \begin{pmatrix} 0 & .3 \\ 0 & 0 \end{pmatrix} \right\rangle$

$= \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & .7 \\ 0 & 0 \end{pmatrix} \right\rangle$

$\Rightarrow$ G+H $\notin$ A{4}

$\Rightarrow$ G+H $\notin$ $A_G\{1,4\}$.

Theorem 3.14:
Let A be symmetric idempotent matrix in $(IF)_n$. Then $A^+ = A$.

Proof:
By Theorem (3.5) and Theorem (3.10),
$A^{(1,3)} = A+K$ where $A \geq AK$ and $A^{(1,4)} = A+H$ where $A \geq HA$ …….(3.7)

By (2.3), $A+AK = A = A+HA$ (3.8)

By Theorem (3.9), for some $A^{(1,3)}$ and $A^{(1,4)}$ inverses of A.

$$\begin{aligned} A^{+} &= A^{(1,4)}A\,A^{(1,3)} \\ &= (A+H)\,A\,(A+K) \\ &= (A^{2} + HA)\,(A+K) \\ &= (A + HA)\,(A+K) \\ &= A\,(A+K) \\ &= A^{2} + AK \\ &= A + AK \\ &= A \end{aligned}$$

$\Rightarrow A^{+} = A.$

Hence the theorem.

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