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Properties of Semi-Symmetric Metric T-connection in an almost contact metric manifolds

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Abstract

In this paper, we have studied the properties of a Semi-symmetric metric T-connection in an almost contact metric manifolds.

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1.Introduction

The idea of a semi-symmetric metric connection on a Riemannian manifold was initiated by Yano[6]. He proved that a Riemannian metric admits a semi-symmetric metric connection whose curvature tensor vanishes, it is necessary and sufficient that the Riemannian metric be conformally flat. Various properties of such connection have been studied by Sharfuddin and Hussain [1], Imai [15], Pathak and De [5], Barua and Ray [4], Pandey and Chaubey [7] and many other geometors. Mishra and Pandey [9], defined a semi-symmetric metric T-connection and studied some properties of almost Grayan and Sasakian manifolds. Semi-symmetric metric T-connection in an almost contact metric manifold is studied by Chaubey and Kumar [14],. In this paper we studied the properties of a semi-symmetric metric T-connection in an almost contact metric manifolds.

2.Preliminaries

Let M^n ($n = 2m + 1$), be an odd dimensional differentiable manifold of differentiability class C^{r+1}, there exist a vector valued real linear function φ, a 1-form η, the associated vector field ξ and the Riemannian metric g satisfying

$$\varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\varphi X) = 0, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for arbitrary vector fields X and Y, then (M^n, g) is said to be an almost contact metric manifold and the structure $\{\varphi, \xi, \eta, g\}$ is called an almost contact metric structure to M^n [11].

In view of (2.1), (2.2) and (2.3), we find

$$\eta(\xi) = 1, \quad g(X,\xi) = \eta(X), \qquad \varphi\,\xi = 0 \tag{2.4}$$

Projective curvature tensor P and Conformal curvature tensor C of the manifold (M^n, g) are given by (Mishra, 1984)

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y], \tag{2.5}$$

and $C(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)RX - g(X,Z)RY]$

$$+\frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \tag{2.6}$$

Divergence of the curvature tensor of a Riemannian manifold (M^n, g) is given by (Mishra, 1984)

$$(div\,R)(X,Y)Z = (D_X Ric)(Y,Z) - (D_Y Ric)(X,Z). \tag{2.7}$$

A Riemannian manifold (M^n, g) is called R-harmonic if $(div\,R)(X,Y)Z = 0$ (Arslan, Ezentas, Murathan and Ozgur, 2001 [16]).

Divergence of the Conformal curvature tensor C is given by (Arslan, Ezentas, Murathan and Ozgur, 2001 [16])

$$(div\,C)(X,Y)Z = \left(\frac{n-3}{n-2}\right)[(D_X Ric)(Y,Z) - (D_Y Ric)(X,Z)]$$

$$-\frac{\mathbf{1}}{\mathbf{2(n-1)}}[g(Y,Z)dr(X) - g(X,Z)dr(Y]. \tag{2.8}$$

A Riemannian manifold is said to be conformally conservative (Chaki and Maithy, 2000 [17]) if

$$(div\,C)(X,Y)Z = 0.$$

3.Semi-Symmetric metric T-connection

Let D be a Riemannian connection, then a linear connection ∇ defined as

$$\nabla_X Y = D_X Y + \pi(Y)X - g(X,Y)\rho \tag{3.1}$$

for arbitrary vector fields X and Y, where π is any 1-form associated with the vector field ρ, i.e.

$$\pi(Y) = g(Y,\rho), \tag{3.2}$$

is called a semi-symmetric metric connection. The torsion tensor S of the connection ∇ and metric tensor g are given by [6]

$$S(X,Y) = \pi(Y)X - \pi(X)Y \tag{3.3}$$

And

$$\nabla_X g = 0 \tag{3.4}$$

Agreement: The manifold (M^n, g) is considered to be an almost contact metric manifold. The equations (3.1), (3.2) and (3.3) becomes

$$\nabla_X Y = D_X Y + \eta(Y)X - g(X,Y)\xi, \tag{3.5}$$

$$\eta(Y) = g(Y,\xi) \tag{3.6}$$

$$S(X,Y) = \eta(Y)X - \eta(X)Y \tag{3.7}$$

If in addition (a) $D_X\xi = 0$ or (b) $(D_X\eta)(Y) = 0$ (3.8)

hold for arbitrary vector fields X and Y, then the connection ∇ is said to be a semi-symmetric metric T-connection [9].

Also from (3.5) and (3.8), we have

$$D_X\xi + X - \eta(X)\xi = 0 \Leftrightarrow (D_X\eta)(Y) + g(\varphi X, \varphi Y) = 0 \tag{3.9}$$

4. Existence of a semi-symmetric metric T-connection

Let X and Y be any two vector fields on (M^n, g). Let us define a connection $D_X Y$ by the following equation:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &+g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \\ &+g(\eta(Y)X - \eta(X)Y, Z) + g(\eta(Y)Z - \eta(Z)Y, X) \\ &+g(\eta(X)Z - \eta(Z)X, Y) \end{aligned} \tag{4.1}$$

It can be easily verified that the mapping

$(X,Y) \to \nabla_X Y$ satisfies the following equalities:

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z \tag{4.2}$$

$$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z \tag{4.3}$$

$$\nabla f_X Y = f\nabla_X Y \tag{4.4}$$

$$\nabla_X(fY) = f\nabla_X Y + (Xf)(Y) \tag{4.5}$$

for all $X, Y, Z \in T(M)$ and $f \in F(M)$, the set of all differentiable mappings over M. From (4.2), (4.3), (4.4) and (4.5) we can conclude that ∇ determine a linear connection on (M^n, g).

Now we have

$$2g(\nabla_X Y, Z) - 2g(\nabla_Y X, Z) = 2g([X,Y],Z)$$

$+2g(\eta(Y)X - \eta(X)Y, Z).$ (4.6)

Hence,
$\nabla_X Y - \nabla_Y X - [X, Y] = \eta(Y)X - \eta(X)Y$

or,
$S(X, Y) = \eta(Y)X - \eta(X)Y$ (4.7)

Also, we have
$2g(\nabla_X Y, Z) + 2g(\nabla_X Z, Y) = 2Xg(Y, Z),$
$(\nabla_X g)(Y, Z) = 0,$ (4.8)

from (4.7) and (4.8) it follows that ∇ determines a semi-symmetric metric T-connection on (M^n, g). It can be easily verified that ∇ determines a unique semi-symmetric metric T-connection on (M^n, g). Hence, we can state the following theorem:

Theorem 4.1 Let M^n be a Riemannian manifold and η be a 1-form on it. Then there exit a unique linear connection ∇ satisfying (4.7)and (4.8).

Remark: The above theorem proves the existence of a semi-symmetric metric T-connection on (M^n, g).

5. Curvature tensor of an almost contact metric manifold with respect to the semi-symmetric metric T-connection

The curvature tensor of the semi-symmetric metric T-connection ∇ is given by

$$\tilde{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and that of Riemannian connection D

$$R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

are related as [9]
$\tilde{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y$ (5.1)

Contracting above equation with respect to X, we have
$\widetilde{Ric}(Y, Z)X = Ric(Y, Z) + (n-1)g(Y, Z),$ (5.2)

where $\widetilde{Ric}$ and Ric are the Ricci – tensor of the connection ∇ and D respectively.
Again contracting (5.2), we get
$\tilde{r} = r + n(n-1),$ (5.3)

where $\tilde{r}$ and r are the scalar curvature tensor of the connection ∇ and D respectively.

Theorem 5.1 In an almost contact metric manifold, Bianchi`s first identity corresponding to the semi-symmetric metric T-connection is given by

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0.$$

Proof: Writing two more equations by the cyclic permutations of X, Y and Z from the equation (5.1) and adding then to (5.1), we get the desired result.

Theorem 5.2 In an almost contact metric manifold M^n admitting a semi-symmetric metric T-connection, we have

$$'\tilde{R}(X,Y,Z,W) + '\tilde{R}(Y,X,Z,W) = 0,$$
$$'\tilde{R}(X,Y,Z,W) + '\tilde{R}(X,Y,W,Z) = 0,$$
$$'\tilde{R}(X,Y,Z,W) + '\tilde{R}(Z,W,X,Y) = 0.$$

Proof: The result is obvious from equation (5.1).

Theorem 5.3 If in an almost contact metric manifold, Ricci tensor of the semi-symmetric metric T-connection vanishes then the manifold is an Einstein manifold.

Proof: If $\widetilde{Ric}(X,Y) = 0$, then from (5.2), it follows that

$$\mathrm{Ric}(X,Y) = -(n-1)g(Y,Z),$$

Which proves the statement.

Theorem 5.4 If in an almost contact metric manifold, Ricci tensor of the semi-symmetric metric T-connection ∇ vanishes, then the manifold is R-harmonic and Conformally conservative.

Proof: If $\widetilde{Ric}(Y,Z) = 0$, then from (5.2), we have

$$\mathrm{Ric}(Y,Z) = -(n-1)g(Y,Z),$$

From which, we get

$$(D_X\mathrm{Ric})(Y,Z) = 0. \tag{5.4}$$

Also, from equation (5.3), we get

$$r = -n(n-1).$$

Hence,

$$dr(X) = 0. \tag{5.5}$$

Now, using (5.4) and (5.5) in (2.7) and (2.8)
We obtain the required result.

6. Projective curvature tensor of a semi-symmetric metric T-connection in an almost contact metric manifold

Projective curvature tensor of an almost contact metric manifold with respect to semi-symmetric metric connection is given by (Mishra, 1984)

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-1)}[\widetilde{Ric}(Y,Z)X - \widetilde{Ric}(X,Z)Y] \tag{6.1}$$

Using (5.1) and (5.2) in (6.1), we get

$$\tilde{P}(X,Y)Z = P(X,Y)Z, \tag{6.2}$$

where $P(X,Y)Z$ is the Projective curvature tensor of an almost contact metric manifold M^n with respect to the Levi-Civita connection.

Hence, we can state the following:

Theorem 6.1 In an almost contact metric manifold M^n, with semi-symmetric metric T-connection, the Projective curvature tensor of the connection ∇ and D are equal.

Theorem 6.2 In an almost contact metric manifold M^n, with semi-symmetric metric T-connection ∇, we have

$$\tilde{P}(X,Y)Z + \tilde{P}(Y,Z)X + \tilde{P}(Z,X)Y = P(X,Y)Z + P(Y,Z)X + P(Z,X)Y.$$

Proof: Writing two more equations by cyclic permutation of X, Y and Z from the equation (6.2) and adding them to (6.2), we get the required result.

Theorem 6.3 If in an almost contact metric manifold M^n, the curvature tensor of semi-symmetric metric T-connection is equal to the Projective curvature tensor of the Levi-Civita connection then the manifold is an Einstein manifold.

Proof: Let $\tilde{R}(X,Y)Z = P(X,Y)Z,$

then using (5.1) in above, we get

$$R(X,Y)Z + g(Y,Z)X - g(X,Z)Y = R(X,Y)Z - \frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y]$$

This gives,

$$g(Y,Z)X - g(X,Z)Y = -\frac{1}{(n-1)}[Ric(Y,Z)X - Ric(X,Z)Y].$$

Putting $X = W = e_i$ in the above equation and taking summation over i, $1 \leq i \leq n$, we get

$$Ric(Y,Z) = -(n-1)g(Y,Z).$$

This shows that the manifold is an Einstein manifold.

7. Conformal curvature tensor of the semi-symmetric metric T-connection on an almost contact metric manifold

Conformal curvature tensor $\tilde{C}$ of an almost contact metric manifold relative to the semi-symmetric metric T-connection ∇ is given by

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-2)}\left[\widetilde{Ric}(Y,Z)X - \widetilde{Ric}(X,Z)Y + g(Y,Z)\tilde{R}X - g(X,Z)\tilde{R}Y\right] + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \tag{7.1}$$

Using (5.1), (5.2)and (5.3) in (7.1), we get

$$\tilde{C}(X,Y)Z = C(X,Y)Z \tag{7.2}$$

where $C(X,Y)Z$ is the Conformal curvature tensor of an almost contact metric manifold with respect to the Levi-Civita connection D.

Theorem 7.1 On an almost contact metric manifold with a semi-symmetric metric T-connection ∇, we have

$$\tilde{C}(X,Y)Z + \tilde{C}(Y,Z)X + \tilde{C}(Z,X)Y = C(X,Y)Z + C(Y,Z)X + C(Z,X)Y.$$

Proof: Writing two more equations from equation (7.2) by the cyclic permutation of X, Y and Z and adding them to (7.2), we get the required result.

Theorem 7.2 If on an almost contact metric manifold, the curvature tensor of the semi-symmetric metric T-connection is equal to the Conformal curvature tensor of the Levi-Civita connection, then M^n is an Einstein manifold.

Proof: If $\tilde{R}(X,Y)Z = C(X,Y)Z$.

Then,

$$'\tilde{R}(X,Y,Z,W) = 'C(X,Y,Z,W) \tag{7.3}$$

where $'C(X,Y,Z,W) = g(\tilde{C}(X,Y)Z,W)$.

Now, using equation (5.1) in (7.3), we get

$$g(Y,Z)g(X,W) - g(X,Z)g(Y,W) = -\frac{1}{(n-2)}\begin{bmatrix} Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W) \\ +g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W)\end{bmatrix} + \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].$$

Now, putting $X = W = e_i$, in the above equation and taking summation over i, $1 \leq i \leq n$, we get

$$Ric(Y,Z) = -(n-1)g(Y,Z),$$

which proves the statement.

Theorem7.3 If on an almost contact metric manifold, the Ricci tensor of the semi-symmetric metric T-connection vanishes, then the projective curvature tensor and the conformal curvature tensor of Levi-Civita connection are equal.

Proof: If $\widetilde{Ric}(Y, Z) = 0$, then from equation (6.1) and (7.1), we get

$$\tilde{R}(X, Y)Z = \tilde{P}(X, Y)Z = \tilde{C}(X, Y)Z.$$

Now, using the above equations in (6.2) and (7.2), we get

$$\tilde{R}(X, Y)Z = P(X, Y)Z = C(X, Y)Z.$$

Thus, from the above equations, we have

$$P(X, Y)Z = C(X, Y)Z.$$

which proves the theorem.

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