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# Growth Rates of Wronskians on the basis of Zero Order

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**Abstract**

In this paper we study the growth properties of composite entire and meromorphic functions which improve some earlier results.

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## 1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let $f$ be a meromorphic function and $g$ be an entire function defined on $\mathbb{C}$. We use the standard notations and definitions in the theory of entire and meromorphic functions which are available in [5] and [13]. In the sequel we use the following notation:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and} \\ \log^{[0]} x &= x. \end{aligned}$$

The following definition is well known.

**Definition 1.1.** The order $\rho_f$ and lower order $\lambda_f$ of an entire function $f$ are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log r}.$$

If $f$ is meromorphic then

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If $\rho_f < \infty$ then $f$ is of finite order. Also $\rho_f = 0$ means that $f$ is of order zero. In this connection Liao and Yang [8] gave the following definition:

**Definition 1.2. [8]** Let $f$ be a meromorphic function of order zero. Then the quantities $\rho_f^*$ and $\lambda_f^*$ of a meromorphic function $f$ are defined as:

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log^{[2]} r} \text{ and } \lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log^{[2]} r}.$$

If $f$ is entire, then

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r} \text{ and } \lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r}.$$

Datta and Biswas [4] gave an alternative definition of zero order and zero lower order of a meromorphic function in the following way:

**Definition 1.3. [4]** Let $f$ be a meromorphic function of order zero. Then the quantities $\rho_f^{**}$ and $\lambda_f^{**}$ of $f$ are defined by:

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{T(r, f)}{\log r} \text{ and } \lambda_f^{**} = \liminf_{r\to\infty} \frac{T(r, f)}{\log r}.$$

If $f$ is an entire function then clearly

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log M(r, f)}{\log r} \text{ and } \lambda_f^{**} = \liminf_{r\to\infty} \frac{\log M(r, f)}{\log r}.$$

**Definition 1.4.** The type $\sigma_f$ and lower type $\bar{\sigma}_f$ of a meromorphic function $f$ are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}} \quad \text{and} \quad \bar{\sigma}_f = \liminf_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

When $f$ is entire, it can be easily verified that

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}} \quad \text{and} \quad \bar{\sigma}_f = \liminf_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

**Definition 1.5. [12]** A function $\rho_f(r)$ is called a proximate order of $f$ relative to $T(r,f)$ if
$(i)$ $\rho_f(r)$ is non-negative and continuous for $r \geqslant r_0$, say,
$(ii)$ $\rho_f(r)$ is differentiable for $r \geqslant r_0$ except possibly at isolated points at which $\rho'_f(r-0)$ and $\rho'_f(r+0)$ exist,
$(iii)$ $\lim_{r\to\infty} \rho_f(r) = \rho_f < \infty$,
$(iv)$ $\lim_{r\to\infty} r\rho'_f(r)\log r = 0$ and
$(v)$ $\limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f(r)}} = 1$.

In the line of Definition 5 the following definition can be given:

**Definition 1.6.** A function $\lambda_f(r)$ is called a lower proximate order of $f$ relative to $T(r,f)$ if
$(i)$ $\lambda_f(r)$ is non-negative and continuous for $r \geqslant r_0$, say,
$(ii)$ $\lambda_f(r)$ is differentiable for $r \geqslant r_0$ except possibly at isolated points at which $\lambda'_f(r-0)$ and $\lambda'_f(r+0)$ exist,
$(iii)$ $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$,
$(iv)$ $\lim_{r\to\infty} r\lambda'_f(r)\log r = 0$ and
$(v)$ $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1$.

The following definitions are also well known.

**Definition 1.7.** A meromorphic function $a = a(z)$ is called small with respect to $f$ if $T(r,a) = S(r,f)$.

**Definition 1.8.** Let $a_1, a_2, \ldots, a_k$ be linearly independent meromorphic functions and small with respect to $f$ .We denote by $L(f) = W(a_1, a_2, \ldots, a_k, f)$ the Wronskian determinant of $a_1, a_2, \ldots, a_k, f$ i.e.,

$$L(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f^{'} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}.$$

**Definition 1.9.** If $a \in \mathbb{C} \cup \{\infty\}$,the quantity

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a; f)}{T(r, f)}$$

is called the Nevanlinna's deficiency of the value $'a'$.

From the second fundamental theorem it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf [5],.$p$.43 ). If in particular, $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that $f$ has the maximum deficiency sum.

In the paper we establish some newly developed results based on the comparative growth properties of composite entire or meromorphic functions and wronskians generated by one of the factors improving some earlier results.

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 2.1. [1]** Let $f$ be meromorphic and $g$ be entire then for all sufficiently large values of $r$,

$$T(r, f \circ g) \leqslant \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

**Lemma 2.2. [2]** Let $f$ be meromorphic and $g$ be entire and suppose that $0 < \mu < \rho_g \leq \infty$. Then for a sequence of values of $r$ tending to infinity,

$$T(r, f \circ g) \geq T(\exp(r^{\mu}), f).$$

**Lemma 2.3. [3]** If $f$ and $g$ are two entire functions then for all sufficiently large values of $r$,

$$M(r, f \circ g) \leq M(M(r, g), f).$$

**Lemma 2.4. [6]** Let $g$ be an entire function with $\lambda_g < \infty$ and assume that $a_i(i = 1, 2, \ldots, n; n \leq \infty)$ are entire functions satisfying $T(r, a_i) = o\{T(r, g)\}$.If $\sum_{i=1}^{n} \delta(a_i, g) = 1$,then $\underset{r\to\infty}{lim} \dfrac{T(r, g)}{\log M(r, g)} = \dfrac{1}{\pi}$.

**Lemma 2.5. [7]** If $f$ be an entire function, then for $\delta( > 0)$ the function $r^{\rho_f + \delta - \rho_{f(r)}}$ is ultimately an increasing function of $r$.

**Lemma 2.6. [9]** Let $f$ be an entire function. Then for $\delta( > 0)$ the function $r^{\lambda_f + \delta - \lambda_{f(r)}}$ is ultimately an increasing function of $r$.

**Lemma 2.7. [10]** Let $f$ be a transcendental meromorphic function having the maximum deficiency sum. Then

$$\lim_{r\to\infty} \frac{T\,(r, L\,(f))}{T\,(r, f)} = 1 + k - k\delta\,(\infty; f)\,.$$

**Lemma 2.8.** If $f$ be a transcendental meromorphic function with the maximum deficiency sum, then the order and lower order of $L(f)$ are same as those of $f$.Also the type and lower type of $L[f]$ is $\{1 + k - k\delta\,(\infty; f)\}$ times that of $f$, i.e., $\rho_{L[f]} = \rho_f$, $\lambda_{L[f]} = \lambda_f$, $\sigma_{L[f]} = \{1 + k - k\delta\,(\infty; f)\,)\}\sigma_f$, and $\bar{\sigma}_{L[f]} = \{1 + k - k\delta\,(\infty; f)\,)\}\bar{\sigma}_f$ when $f$ is of finite positive order.

*Proof.* By Lemma 2.7,

$$\lim_{r\to\infty} \frac{\log T(r, L(f))}{\log T(r, f)}$$

exists and is equal to 1. So

$$\begin{aligned}
\rho_{L[f]} &= \limsup_{r\to\infty} \frac{\log T(r, L(f))}{\log r} \\
&= \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \cdot \lim_{r\to\infty} \frac{\log T(r, L(f))}{\log T(r, f)} \\
&= \rho_f.1 = \rho_f.
\end{aligned}$$

In a similar manner, $\lambda_{L[f]} = \lambda_f$.
Again

$$\begin{aligned}
\sigma_{L[f]} &= \limsup_{r\to\infty} \frac{T\,(r, L[f])}{r^{\rho_{L[f]}}} \\
&= \lim_{r\to\infty} \frac{T\,(r, L[f])}{T\,(r, f)} \cdot \limsup_{r\to\infty} \frac{T\,(r, f)}{r^{\rho_f}} \\
&= \{1 + k - k\delta\,(\infty; f)\}\sigma_f.
\end{aligned}$$

Similarly $\bar{\sigma}_{L[f]} = \{1 + k - k\delta\,(\infty; f)\,)\}\bar{\sigma}_f$. This proves the lemma. ■

## 3. Theorems

In this section we present the main results of the paper.

**Theorem 3.1.** Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be an entire function such that $(i)$ $0 < \rho_f < \infty$, $(ii)$ $\rho_f = \rho_g$, $(iii)$ $\bar{\sigma}_f > 0$ and $(iv)$ $\sigma_g < \infty$. Then

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, L(f))} &\leq \rho_f\frac{\sigma_g}{\{1+k-k\delta(\infty; f)\}\bar{\sigma}_f}\\ &= \rho_g\frac{\sigma_g}{\{1+k-k\delta(\infty; f)\}\bar{\sigma}_f}.\end{aligned}$$

*Proof.* As $T(r,g) \leqslant \log^+ M(r,g)$, we have from Lemma 2.1 for all sufficiently large values of $r$,

$$\begin{aligned}T(r, f\circ g) &\leqslant \{1+o(1)\}T(M(r,g), f)\\ i.e.,\ \log T(r, f\circ g) &\leqslant \left(\rho_f+\varepsilon\right)\log M(r,g)+O(1)\\ i.e.,\ \limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r,f)} &\leqslant \left(\rho_f+\varepsilon\right)\limsup_{r\to\infty}\frac{\log M(r,g)}{T(r,f)}. \qquad (1)\end{aligned}$$

Also from the definitions of type and lower type it follows for all sufficiently large values of $r$

$$\log M(r,g) \leq \left(\sigma_g+\varepsilon\right) r^{\rho_g} \qquad (2)$$

and

$$T(r,f) \geq \left(\bar{\sigma}_f-\varepsilon\right) r^{\rho_f}. \qquad (3)$$

Since $\rho_f = \rho_g$ we get from (2) and (3) that

$$\limsup_{r\to\infty}\frac{\log M(r,g)}{T(r,f)} \leq \frac{\sigma_g}{\bar{\sigma}_f}. \qquad (4)$$

As $\varepsilon\ (>0)$ is arbitrary and $\rho_f = \rho_g$, from (1) and (4) we obtain that

$$\limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r,f)} \leq \rho_f\frac{\sigma_g}{\bar{\sigma}_f} = \rho_g\frac{\sigma_g}{\bar{\sigma}_f}. \qquad (5)$$

Now in view of (5) and Lemma 2.7 we get that

$$\begin{aligned}\limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, L(f))} &= \limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r,f)}.\lim_{r\to\infty}\frac{T(r,f)}{T(r, L(f))}\\ &\leq \rho_f\frac{\sigma_g}{\{1+k-k\delta(\infty; f)\}\bar{\sigma}_f}\\ &= \rho_g\frac{\sigma_g}{\{1+k-k\delta(\infty; f)\}\bar{\sigma}_f}.\end{aligned}$$

This completes the proof. ■

The following theorem can be proved in the line of Theorem 3.1 and so the proof is omitted.

**Theorem 3.2.** Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be an entire function such that $(i)$ $0 < \rho_f < \infty$, $(ii)$ $\rho_f = \rho_g$, $(iii)$ $\sigma_f > 0$ and $(iv)$ $\sigma_g < \infty$. Then

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} &\le \rho_f \frac{\sigma_g}{\{1 + k - k\delta(\infty; f)\}\sigma_f} \\ &= \rho_g \frac{\sigma_g}{\{1 + k - k\delta(\infty; f)\}\sigma_f}.\end{aligned}$$

In the line of Theorem 3.2 the following two corollaries can also be proved:

**Corollary 3.3.** Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be an entire function such that $(i)$ $0 < \rho_f < \infty$, $(ii)$ $\rho_f = \rho_g$, $(iii)$ $\overline{\sigma}_f > 0$ and $(iv)$ $\overline{\sigma}_g < \infty$. Then

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} &\le \rho_f \frac{\overline{\sigma}_g}{\{1 + k - k\delta(\infty; f)\}\overline{\sigma}_f} \\ &= \rho_g \frac{\overline{\sigma}_g}{\{1 + k - k\delta(\infty; f)\}\overline{\sigma}_f}.\end{aligned}$$

**Corollary 3.4.** Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be an entire function such that $(i)$ $0 < \lambda_f < \infty$, $(ii)$ $\rho_f = \rho_g$, $(iii)$ $\overline{\sigma}_f > 0$ and $(iv)$ $\sigma_g < \infty$. Then

$$\liminf_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(f))} \le \lambda_f \frac{\sigma_g}{\{1 + k - k\delta(\infty; f)\}\overline{\sigma}_f}.$$

We may now state the following corollary without proof for the right factor $g$ of the composite function $f \circ g$ based on the type of an entire function:

**Corollary 3.5.** Let $f$ be a meromorphic function and $g$ be a transcendental entire function such that $(i)$ $0 < \rho_f < \infty$, $(ii)$ $0 < \lambda_g \le \rho_g < \infty$, $(iii)$ $0 < \overline{\sigma}_g \le \sigma_g < \infty$ and $(iv)$ $\sum_{a\neq\infty} \delta(a; g) + \delta(\infty; g) = 2$. Then

$$\limsup_{r\to\infty} \frac{\log T(r, f \circ g)}{T(r, L(g))} \le \rho_f \frac{\sigma_g}{\{1 + k - k\delta(\infty; f)\}\overline{\sigma}_g}.$$

The proof is omitted as it can be carried out in the line of Theorem 3.1. ■

**Remark 3.6.** Under the same conditions of Corollary 3.5 if $f$ be a meromorphic function with order zero and $0 < \rho_f^{**} < \infty$ then by Definition 1.3 and in the line of Theorem 3.1 one can easily verify that

$$\limsup_{r\to\infty}\frac{T(r, f\circ g)}{T(r, L(g))} \leq \{1+o(1)\}\rho_f^{**}\frac{\sigma_g}{\{1+k-k\delta(\infty; f)\}\bar{\sigma}_g}.$$

**Remark 3.7.** If we take " $0 < \lambda_f < \infty$" instead of " $0 < \rho_f < \infty$" in Corollary 3.5 and the other conditions remain the same then it can be shown in the line of Corollary 3.4 that

$$\liminf_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, L(g))} \leq \lambda_f\frac{\sigma_g}{\{1+k-k\delta(\infty; f)\}\bar{\sigma}_g}.$$

**Remark 3.8.** In Remark 3.6, if we take " $0 < \lambda_f^{**} < \infty$" instead of " $0 < \rho_f^{**} < \infty$" and the other conditions remain the same then it can be shown that

$$\liminf_{r\to\infty}\frac{T(r, f\circ g)}{T(r, g)} \leq \{1+o(1)\}\lambda_f^{**}\frac{\sigma_g}{\{1+k-k\delta(\infty; f)\}\bar{\sigma}_g}.$$

**Theorem 3.9.** Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be an entire function such that $0 < \rho_f < \rho_g < \infty$. Then

$$\limsup_{r\to\infty}\frac{\log T(r, f\circ g)}{T(r, L(f))} = \infty.$$

*Proof.* Since $\rho_f < \rho_g$ we can choose $\varepsilon\,(> 0)$ in such a way that

$$\rho_f + \varepsilon < \rho_g - \varepsilon < \rho_g\,. \tag{6}$$

Now we obtain from Lemma 2.2 for a sequence of values of $r$ tending to infinity that

$$\begin{aligned}
\log T(r, f\circ g) &\geq \log T\left\{\exp r^{(\rho_g-\varepsilon)}, f\right\} \\
i.e.,\ \log T(r, f\circ g) &\geq \left(\lambda_f-\varepsilon\right)\log\exp r^{(\rho_g-\varepsilon)} \\
i.e.,\ \log T(r, f\circ g) &\geq \left(\lambda_f-\varepsilon\right) r^{(\rho_g-\varepsilon)}.
\end{aligned} \tag{7}$$

Again in view of Lemma 2.8 we have for all sufficiently large values of $r$ that

$$\begin{aligned}
\log T(r, L(f)) &\leq \left(\rho_{L(f)}+\varepsilon\right)\log r \\
i.e.,\ \log T(r, L(f)) &\leq \left(\rho_f+\varepsilon\right)\log r \\
i.e.,\ T(r, L(f)) &\leq r^{(\rho_f+\varepsilon)}.
\end{aligned} \tag{8}$$

Therefore from (7) and (8) it follows for a sequence of values of $r$ tending to infinity,

$$\frac{\log T\,(r, f \circ g)}{T\,(r, L(f))} \geq \frac{\left(\lambda_f - \varepsilon\right) r^{(\rho_g - \varepsilon)}}{r^{(\rho_f + \varepsilon)}}. \tag{9}$$

Now in view of (6) it follows from (9) that

$$\limsup_{r\to\infty} \frac{\log T\,(r, f \circ g)}{T\,(r, L(f))} = \infty.$$

This proves the theorem. ■

In the line of Theorem 3.9 the following corollary may be deduced:

**Corollary 3.10.** Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be an entire function such that $0 < \rho_f < \lambda_g < \infty$. Then

$$\limsup_{r\to\infty} \frac{\log T\,(r, f \circ g)}{T\,(r, L(f))} = \infty.$$

**Theorem 3.11.** Let $f$ be a meromorphic function of order zero and $g$ be a transcendental entire function of non zero finite order satisfying $\sum\limits_{a\neq\infty} \delta\,(a; g) + \delta\,(\infty; g) = 2$, then

$$\limsup_{r\to\infty} \frac{\log T\,(r, f \circ g)}{\log T\,(r, L(g))} \geq 1.$$

*Proof.* In view of Lemma 2.2 we obtain for a sequence of values of $r$ tending to infinity that

$$\begin{aligned}
T\,(r, f \circ g) &\geq T\left\{\exp r^{(\rho_g - \varepsilon)}, f\right\} \\
i.e.,\ T\,(r, f \circ g) &\geq \left(\lambda_f^{**} - \varepsilon\right) \log \exp r^{(\rho_g - \varepsilon)} \\
i.e.,\ T\,(r, f \circ g) &\geq \left(\lambda_f^{**} - \varepsilon\right) r^{(\rho_g - \varepsilon)} \\
i.e.,\ \log T\,(r, f \circ g) &\geq \left(\rho_g - \varepsilon\right) \log r + O(1).
\end{aligned} \tag{10}$$

Again for all sufficiently large values of $r$, we obtain by Lemma 2.7,

$$\log T\,(r, L(g)) \leq \left(\rho_{L(g)} + \varepsilon\right) \log r = \left(\rho_g + \varepsilon\right) \log r\ . \tag{11}$$

Therefore from (10) and (11) we obtain for a sequence of values of $r$ tending to infinity that

$$\frac{\log T\,(r, f \circ g)}{\log T\,(r, L(g))} \geq \frac{\left(\rho_g - \varepsilon\right) \log r + O(1)}{\left(\rho_g + \varepsilon\right) \log r}. \tag{12}$$

Since $\varepsilon\ (>0)$ is arbitrary, the theorem follows from (12) . ■

**Theorem 3.12.** Let $f$ be a meromorphic function of order zero and $g$ be transcendental entire such that $\rho_g$ is finite and $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then for any $\alpha>1$,

$$\liminf_{r\to\infty}\frac{T(r,f\circ g)}{T(r,L(g))}\leq\frac{(1+o(1))(\alpha+1).\rho_f^{**}.(\alpha)^{\rho_g}}{(\alpha-1)(1+k-k\delta(\infty;g))}.$$

*Proof.* If $\rho_f^{**}=\infty$, then the result is obvious. So we suppose that $\rho_f^{**}<\infty$. Since $T(r,g)\leq\log^+ M(r,g)$, we obtain by Lemma 2.1 for $\varepsilon\ (>0)$ and for all sufficiently large values of $r$,

$$T(r,f\circ g)\leq(1+o(1))\left(\rho_f^{**}+\varepsilon\right)\log M(r,g)$$

Since $\varepsilon\ (>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}\leq(1+o(1))\,\rho_f^{**}\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}. \tag{13}$$

Since $\limsup_{r\to\infty}\frac{T(r,g)}{r^{\rho_g(r)}}=1$, for given $\varepsilon\ (0<\varepsilon<1)$ we get for all sufficiently large values of $r$,

$$T(r,g)<(1+\varepsilon)\,r^{\rho_g(r)} \tag{14}$$

and for a sequence of values of $r$ tending to infinity

$$T(r,g)>(1-\varepsilon)\,r^{\rho_g(r)}. \tag{15}$$

Since $\log M(r,g)\leq\left(\frac{\alpha+1}{\alpha-1}\right)T(\alpha r,g)$ $\{cf.[5]\}$, for a sequence of values of $r$ tending to infinity we get for any $\delta\ (>0)$

$$\begin{aligned}\frac{\log M(r,g)}{T(r,g)}&\leq\frac{(\alpha+1)(1+\varepsilon)}{(\alpha-1)(1-\varepsilon)}.\frac{(\alpha r)^{\rho_g+\delta}}{(\alpha r)^{\rho_g+\delta-\rho_g(\alpha r)}}.\frac{1}{r^{\rho_g(r)}}\\&\leq\frac{(\alpha+1)(1+\varepsilon)}{(\alpha-1)(1-\varepsilon)}.(\alpha)^{\rho_g+\delta}.\end{aligned}$$

because by Lemma 2.5, $r^{\rho_g+\delta-\rho_{g(r)}}$ is ultimately an increasing function of $r$. Since $\varepsilon\ (>0)$ and $\delta\ (>0)$ are arbitrary, we obtain from above that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}\leq\frac{(\alpha+1)}{(\alpha-1)}.(\alpha)^{\rho_g}. \tag{16}$$

Thus from (13) and (16) we obtain that

$$\liminf_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}\leq(1+o(1))\frac{(\alpha+1)}{(\alpha-1)}.\rho_f^{**}.(\alpha)^{\rho_g}. \quad (17)$$

Now in view of (17) and Lemma 2.7 we get that

$$\begin{aligned}\liminf_{r\to\infty}\frac{T(r,f\circ g)}{T(r,L(g))} &= \liminf_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}.\lim_{r\to\infty}\frac{T(r,g)}{T(r,L(g))}\\ &\leq \frac{(1+o(1))(\alpha+1).\rho_f^{**}.(\alpha)^{\rho_g}}{(\alpha-1)(1+k-k\delta(\infty;g))}.\end{aligned}$$

This proves the theorem. ■

In the line of Theorem 3.12 one can easily prove the following theorem using the definition of lower proximate order:

**Theorem 3.13.** Let $f$ be a meromorphic function of order zero and $g$ be a transcendental entire function with maximum deficiency sum and $\lambda_g<\infty$. Then for any $\alpha>1$,

$$\liminf_{r\to\infty}\frac{T(r,f\circ g)}{T(r,L(g))}\leq\frac{(1+o(1))(\alpha+1).\rho_f^{**}.(\alpha)^{\lambda_g}}{(\alpha-1)(1+k-k\delta(\infty;g))}.$$

**Theorem 3.14.** Let $f$ be a non constant entire function such that $\rho_f^{**}$ is finite and $g$ be a non constant transcendental entire function with the maximum deficiency sum and finite lower order. Also suppose that there exist entire functions $a_i$ $(i=1,2,\ldots,n;n\leq\infty)$ satisfying $(i)$ $T(r,a_i)=o\{T(r,g)\}$ as $r\to\infty$ for $i=1,2,\ldots,n$ $(ii)$ $\sum_{i=1}^{n}\delta(a_i,g)=1$. Then

$$\frac{\pi\lambda_f^{**}}{3.4^{\lambda_g}\{1+k-k\delta(\infty;g)\}}\leq\limsup_{r\to\infty}\frac{T(r,f\circ g)}{T(r,L(g))}\leq\frac{\pi\rho_f^{**}}{\{1+k-k\delta(\infty;g)\}}.$$

*Proof.* Let us choose $\varepsilon$ in such a way that $0<\varepsilon<\min\left\{\lambda_f^{**},1\right\}$. For any two entire functions $f$ and $g$, the following two inequalities are well known,

$$T(r,f)\leq\log^{+}M(r,f)\leq 3T(2r,f).\ \{cf.[5]\} \quad (18)$$

and

$$T(r,f\circ g)\geq\frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+o(1),f\right\}.\ \{cf.[11]\} \quad (19)$$

Now from (19) we get for all sufficiently large values of $r$,

$$\begin{aligned}
T(r, f\circ g) &\geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\log\left\{\frac{1}{8}M\left(\frac{r}{4},g\right)+\circ(1)\right\}\\
i.e.,\ T(r, f\circ g) &\geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\log\left\{\frac{1}{9}M\left(\frac{r}{4},g\right)\right\}\\
i.e.,\ T(r, f\circ g) &\geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\log M\left(\frac{r}{4},g\right)+\frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\log\frac{1}{9}\\
i.e.,\ T(r, f\circ g) &\geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\log M\left(\frac{r}{4},g\right)+O(1)\\
i.e.,\ \frac{T(r, f\circ g)}{T(r,g)} &\geq \frac{1}{3}\left(\lambda_f^{**}-\varepsilon\right)\frac{\log M\left(\frac{r}{4},g\right)}{T\left(\frac{r}{4},g\right)}.\frac{T\left(\frac{r}{4},g\right)}{T(r,g)}+O(1). \qquad (20)
\end{aligned}$$

Since $\liminf\limits_{r\to\infty}\frac{T(r,g)}{r^{\lambda_g(r)}}=1$ for given $\varepsilon\ (>0)$ we get for all sufficiently large values of $r$,

$$T(r,g) > (1-\varepsilon)\, r^{\lambda_g(r)} \qquad (21)$$

and for a sequence of values of $r$ tending to infinity

$$T(r,g) < (1+\varepsilon)\, r^{\lambda_g(r)}. \qquad (22)$$

From (21) and (22) and in the line of Lemma 2.6 we get for a sequence of values of $r$ tending to infinity and for $\delta\ (>0)$

$$\begin{aligned}
\frac{T\left(\frac{r}{4},g\right)}{T(r,g)} &> \frac{1-\varepsilon}{1+\varepsilon}.\frac{\left(\frac{r}{4}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4}\right)}}.\frac{1}{r^{\lambda_g(r)}}\\
&\geq \frac{1-\varepsilon}{1+\varepsilon}.\frac{1}{4^{\lambda_g+\delta}}.
\end{aligned}$$

Since $\varepsilon\ (>0)$ and $\delta\ (>0)$ are arbitrary we get from (20), Lemma 2.4 and above that

$$\limsup_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)} \geq \frac{\pi\lambda_f^{**}}{3.4^{\lambda_g}}. \qquad (23)$$

Now in view of (23) and Lemma 2.7 we get that

$$\begin{aligned}
\limsup_{r\to\infty}\frac{T(r,f\circ g)}{T(r,L(g))} &= \liminf_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}.\lim_{r\to\infty}\frac{T(r,g)}{T(r,L(g))}\\
&\geq \frac{\pi\lambda_f^{**}}{3.4^{\lambda_g}\{1+k-k\delta(\infty;g)\}}. \qquad (24)
\end{aligned}$$

Again in view of inequality (18) we get from Lemma 2.3, Lemma 2.4 and Lemma 2.7 that for all sufficiently large values or $r$,

$$\begin{aligned} T(r, f \circ g) &\leq \log M(M(r,g), f) \\ i.e.,\ T(r, f \circ g) &\leq \left(\rho_f^{**} + \varepsilon\right) \log M(r,g) \\ i.e.,\ \frac{T(r, f \circ g)}{T(r, L(g))} &\leq \left(\rho_f^{**} + \varepsilon\right) \frac{\log M(r,g)}{T(r, L(g))} \\ i.e.,\ \limsup_{r\to\infty} \frac{T(r, f \circ g)}{T(r, L(g))} &\leq \left(\rho_f^{**} + \varepsilon\right) \limsup_{r\to\infty} \frac{\log M(r,g)}{T(r, L(g))} \end{aligned}$$

$$\begin{aligned} i.e.,\ \limsup_{r\to\infty} \frac{T(r, f \circ g)}{T(r, L(g))} &\leq \left(\rho_f^{**} + \varepsilon\right) \limsup_{r\to\infty} \frac{\log M(r,g)}{T(r,g)} \lim_{r\to\infty} \frac{\log T(r,g)}{T(r, L(g))} \\ i.e.,\ \limsup_{r\to\infty} \frac{T(r, f \circ g)}{T(r, L(g))} &\leq \left(\rho_f^{**} + \varepsilon\right) \pi \frac{1}{\{1 + k - k\delta(\infty; g)\}}. \end{aligned}$$

■

Since $\varepsilon\ (> 0)$ is arbitrary it follows from above that

$$\limsup_{r\to\infty} \frac{T(r, f \circ g)}{T(r, L(g))} \leq \frac{\pi \rho_f^{**}}{\{1 + k - k\delta(\infty; g)\}}. \tag{25}$$

Thus the theorem follows from (24) and (25) .

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