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On Generalized Useful Entropy for Incomplete Probability Distribution

P. Jha and Anjali Chandravanshi

Deptt. Of Maths, Govt. Chhattisgarh P.G. College,
Raipur (Chhattisgarh) INDIA

Abstract

In 2005 Khan, Bhat and S. Pirzada proved a noiseless coding theorem by considering useful entropy and useful mean codeword length. In this communication we consider a generalization of the useful mean codeword length and derived lower and upper bounds for it in terms of useful entropy for incomplete probability distribution.

Key Words: Useful entropy, useful mean code word length, coding theorem.

Subject Classification No.: 94A17, 94A24.

INTRODUCTION

Consider the following model for a random experiment S,

$$S_N = [E; P; U] \tag{1.1}$$

where $E = (E_1, E_2, .., E_N)$ is a finite system of events, $P = (p_1, p_2, .., p_N), 0 \le p_i \le 1, \quad \sum p_i = 1$ is the probability distribution and $U = (u_1, u_2 ..., u_N), u_i \ge 0, i = 1,2,..., N$ is the utility distribution. The u_i's are non-negative real numbers. Denote the model by E, where

$$E = \begin{bmatrix} E_1\ E_2 \ldots E_N \\ p_1\ p_2 \cdots p_N \\ u_1\ u_2 \ldots u_N \end{bmatrix}$$

we call (1.1) a utility information scheme. Belis and Guiasu [1] introduced the measure

$$H(P;U) = -\sum u_i p_i \log p_i \tag{1.2}$$

about the scheme (1.1). They called it useful information for this scheme, where $H(P;U)$ reduces to Shannon's [10] entropy when $u_i = 1$ i.e., the utility aspect of the scheme is ignored for each i. Unless otherwise stated $\sum$ will stand for $\sum_{i=1}^{N}$ and the logarithms are to the base D(D>1) throughout the paper. Guiasu and Picard [4] considered the problem of encoding the outcomes in (1.1) by mean of a prefix code with code words $w_1, w_2, \ldots. w_N$ having lengths respectively $n_1, n_2, \ldots. n_N$ and satisfying Kraft's inequality [3]

$$\sum D^{-n_i} \leq 1 \tag{1.3}$$

where D is the size of code alphabate. They introduced the following useful mean length of the code

$$L(U) = \frac{\sum u_i p_i n_i}{\sum u_i p_i} \tag{1.4}$$

and the authors obtained bounds for it in terms of $H(P:U)$.

Longo[8] , Gurdial and Pessoa[5],Khan and Autar[6],Khan ,Bhat and Pirzada[7] have studied generalized coding theorem by considering different generalized measure of (1.2) and (1.4) under condition(1.3) of unique deciferablity.

In this paper, we study upper and lower bound by considering a new function depending on the parameters α and β and a utility function .Our motivation for studing this function is that it generalizes some information measures already existing in the literature.

2. Coding Theorem

Consider a function

$$H_{\alpha,\beta}(P;U) = \left[D^{\frac{\alpha-1}{\alpha}} - 1\right]^{-1}\left[1 - \left(\frac{\sum u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\beta}}\right)^{1/\alpha}\right] \tag{2.1}$$

where $\alpha > 0 (\neq 1), \beta > 0, p_i \geq 0, i = 1,2, \ldots. N$,and $\sum p_i \leq 1$.

(i) when $\beta = 1$ and $\alpha \to 1$, (2.1) reduces to a measure of information due to Belis and Guiasu[1].
(ii) when $u_i = 1$ for each i, i.e., when the utility aspect is ignored $\sum p_i = 1$, $\beta = 1$ and $\alpha \to 1$, the measure (2.1) reduces to Shannon's entropy [10].
(iii) when $u_i = 1$ for each i,the measure (2.1) becomes the entropy for the β-power distribution derived from P studied by Roy[9]. We call $H_{\alpha,\beta}(P;U)$ in (2.1) the generalized useful measure of information for the incomplete power distribution P^{β}.

Further consider

$$L_{\alpha,\beta}(U) = \left[D^{\frac{\alpha-1}{\alpha}} - 1\right]^{-1}\left[1 - \sum p_i^{\beta}\left(\frac{u_i}{\sum u_i p_i^{\beta}}\right)^{1/\alpha} D^{\frac{-(\alpha-1)n_i}{\alpha}}\right] \tag{2.2}$$

(i) For $\beta = 1$, the length (2.2) reduces to the useful mean length of the code given by Bhatia[2].
(ii) For $\beta = 1$ and $\alpha \to 1$, the length (2.2) reduces to the useful mean length $L(U)$ of the code given in (1.4).
(iii)When the utility concept of the scheme is ignored by taking $u_i = 1$ for each i, $\sum p_i = 1$, $\beta = 1$ and $\alpha \to 1$, the mean length becomes optimal code length defined by Shannon [10].

We establish a result, that in sense, provided a characterization of unique decifer-ability.

THEOREM 2.1:- For all integers D>1, let n_i satisfy (1.3), then the generalized average useful code word length satisfies

$$L_{\alpha,\beta}(U) \geq H_{\alpha,\beta}(P;U) \tag{2.3}$$

and the equality holds iff

$$n_i = -\log\left(\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) \tag{2.4}$$

Proof:-
By Holder's inequality [10]

$$\sum x_i y_i \geq \left(\sum x_i^p\right)^{1/p}\left(\sum y_i^q\right)^{1/q} \tag{2.5}$$

For all $x_i, y_i > 0, i = 1,2.....N$ and $\frac{1}{p}+\frac{1}{q}=1, p<1(\neq 0), q<0$ or $q<1(\neq 0), p<0.$

We see that equality holds if and only if there exists a positive constant c such that

$$x_i^p = cy_i^q \quad (2.6)$$

Making the substitution

$$p=\frac{\alpha-1}{\alpha}, q=1-\alpha$$

$$x_i = p_i^{\frac{\alpha\beta}{\alpha-1}}\left[\frac{u_i}{\sum u_i p_i^{\beta}}\right]^{\frac{1}{\alpha-1}} D^{-n_i} \quad , \quad y_i = p_i^{\frac{\alpha\beta}{1-\alpha}}\left[\frac{u_i}{\sum u_i p_i^{\beta}}\right]^{\frac{1}{1-\alpha}}$$

in (2.6), using (1.3) and after making suitable operations we get (2.4) for $\left[D^{\frac{\alpha-1}{\alpha}}-1\right]\neq 0$ according as $\alpha \neq 1.$

It is clear that the equality in (2.4) holds if and only if

$$D^{-n_i} = \frac{u_i p_i^{\alpha\beta}}{\sum u p_i^{\alpha\beta}}$$

or

$$n_i = -\log\left(\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right)$$

THEOREM 2.2:- For every code with length $\{n_i\}, i=1,2,...N$, of theorem 2.1, $L_{\alpha,\beta}(U)$ can be made to satisfy

$$H_{\alpha,\beta}(P;U) \leq L_{\alpha,\beta}(U) < H_{\alpha,\beta}(P;U) D^{\frac{1-\alpha}{\alpha}} + \left[D^{\frac{\alpha-1}{\alpha}}-1\right]^{-1}\left[1-D^{\frac{1-\alpha}{\alpha}}\right] \quad (2.7)$$

Proof:- From (2.4) it can be concluded that it is always possible to have a code satisfying

$$-\log\left(\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right) \leq n_i < -\log\left(\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right)+1$$

or

$$\left[\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right]^{-1} \le D^{n_i} < D\left[\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right]^{-1} \tag{2.8}$$

From (2.8), we have

$$D^{n_i} < D\left[\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right]^{-1}$$

or

$$D^{-n_i\left(\frac{\alpha-1}{\alpha}\right)} < D^{\frac{1-\alpha}{\alpha}}\left[\frac{u_i p_i^{\alpha\beta}}{\sum u_i p_i^{\alpha\beta}}\right]^{\left(\frac{\alpha-1}{\alpha}\right)} \tag{2.9}$$

Multiplying both sides of (2.9) by $p_i^{\beta}\left[\frac{u_i}{\sum u_i p_i^{\beta}}\right]^{1/\alpha}$ summing over $i = 1,2,....N$ and making suitable operation we get ,

$$L_{\alpha,\beta}(U) < H_{\alpha,\beta}(P;U)D^{\frac{1-\alpha}{\alpha}} + \left[D^{\frac{\alpha-1}{\alpha}} - 1\right]^{-1}\left[1 - D^{\frac{1-\alpha}{\alpha}}\right] \tag{2.10}$$

which proves the theorem 2.2.

Remark: For $0 < \alpha < 1$ and since $D \ge 2$ from (2.10), we have $\left[D^{\frac{\alpha-1}{\alpha}} - 1\right]^{-1}\left[1 - D^{\frac{1-\alpha}{\alpha}}\right] > 1$ from which it follows that the upper bound of $L_{\alpha,\beta}(U)$ in (2.7) is grater than unity.

References

[1] M.Belis and S.Guiasu, A qualitative-quantitative measure of information in Cyberna-tics Systems, IEEE Trans. Information Theory, IT-14(1968), 593-594.

[2] P.K.Bhatia, On generalized useful inaccuracy for incomplete probability distribution, Soochow J. Math., 25(2) (1999), 131-135.

[3] A.Feinstein, Foundation of Information Theory, McGraw HILL, New York,(1958).

[4] S. Guiasu and C.F.Picard, Born infericutre de la Longuerur utile de certain codes, C.R. Acad. Sci. Paris, 273A (1971), 248-251.

[5] Gurdial and F.Pessoa, On useful information of order α, J. Comb. Information and syst. Sci., 2(1977), 158-162.

[6] A.B.Khan and R.Autar, On useful information of order α and β, Soochow J.Math., 5(1979),93-99.

[7] A.B.Khan, B.A.Bhat and S,Pirzada,"Some result on a generalized useful information mea-sure",J. Ineq. Pure and Appl.Math., 6(4) Art.117, (2005).

[8] G.Longo, A noiseless coding theorem for sources having utilities, SIAM J. Appl. Math., 30(4) (1979), 739-748.

[9] L.K.Roy, Comparison of Renyi entropies of power distribution, AMM,56(1976),217-218.

[10] C.E.Shannon, A Mathematical theory of Communication, Bell Sysstem, Tech. J., 27(1948), 394-423, 623-656.

[11] O.Shisha, Inequalities, Academic Press, New York (1967).