Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 5, Number 1 (2013), pp. 53-67

http://www.irphouse.com

Invariant Monotonicity (Quasi &Pseudo) and Invexity

[*] S.K. Pradhan, [**] D.K. Dalai and [***] R.B. Dash

** Department of Mathematics, D.A.V. College Koraput-764001, Odisha, India*
E-mail address: sarojkantap@ yahoo. in
*** Reader in Mathematics, S.B. Women's College, cuttack-753001, Odisha, India*
E-mail address:drdalai@yahoo.com
**** Reader in Mathematics, Ravenshaw University, cuttack-753001, Odisha, India*
E-mail address: rajani _bdash@radif mail.com

Abstract

Several kinds of invariant quasi and pseudo monotone maps are introduced. Some examples are given which show that every quasi and pseudo monotone maps are invariant quasi and pseudo monotone maps. Relationships between generalized invariant quasi and pseudo monotonicity and generalized invexity are established.

Our results are generalizations of those presented by X.M.Yang, X.Q Yang and K.L.Teo.

1. Introduction:-

Convexity is a common assumption made in mathematical programming. In recent years, there have been increasing attempts to weaken the convexity condition. Consequently, several classes of (generalized) invex functions have been introduced in the literature. More specifically, the concept of invexity was introduced in Ref [1], where it is shown that the Kuhn-Tucker conditions are sufficient for (global) optimality of nonlinear programming problems under invexity condition. In Ref [2]-[3], Weir and Mond introduced the concept of pre invex functions, and applied it to the establishment of the sufficient optimality conditions and duality in (multiobjective) nonlinear programming. In Ref [4] Mohan and Neogy showed that, under certain conditions, an invex function is preinvex and a quasiinvex function is prequasiinvex. Convexity of a real-valued function is the monotonicity of a vector valued function and convexity of real-valued function is equivalent to the monotonicity of the corresponding gradient function. An important breakthrough generalization of this relation was given in Ref [5] for various psedo/quasiconvexities and psedo/quasimonotonicities.

We introduced several types of generalized invariant monotonicities which are generalization of the (strict) monotonicity, (strict) pseudomonotonicity and quasimonotoniciy mentioned in Ref [5].The main aim of this paper is to establish relations among generalized invariant monotonicities and generalized invexities.Note that the conditions assumed in this paper are different from those assumed in Ref. [6]-[7]. Several examples are given to show that these generalized invariant monotonicities (quasi and pseudo) are proper generalization of the corresponding generalized monotonicities. Moreover, some examples are also presented to illustrate the properly relation among the generalized invariant monotonicities.

In this paper we further generalized the idea of X.M.Yang, X.Q.Yang and K.L.Teo in Ref [8] taking into account of three variables instead of two variables

2. Invariant Quasimonotone Maps

Definition 2.1. See Ref 5. A map F is Quasimonotone on a set Γ of R^n if, for every point x, y, z $\in$ Γ, $(y-x)^T F(x) > 0$ implies $(y-x)^T F(y) \geq 0$

$$(z-y)^T F(y) > 0 \text{ implies } (z-y)^T F(y) \geq 0$$

$$(x-z)^T F(y) > 0 \text{ implies } (x-z)^T F(y) \geq 0 .$$

Definition 2.2. Let Γ of R^n be an invex set with respect to η. A map F is invariant quasimonotone with respect to the same η on Γ if, for every distinct point x, y, z $\in$ Γ there exist $\eta : R^n \times R^n \times R^n \to R^n$ such that

$$\eta(y,x,z)^T F(x) > 0 \text{ implies } \eta(x,y,z)^T F(y) \leq 0$$

$$\eta(x,z,y)^T F(y) > 0 \text{ implies } \eta(x,y,z)^T F(z) \leq 0$$

$$\eta(y,x,z)^T F(z) > 0 \text{ implies } \eta(y,z,x)^T F(x) \leq 0$$

Assumption A. Let the set Γ be invex with respect to η, and let $f : \Gamma \to R$ .Then

$$f\left(z+\eta(x,y,z)\right) \leq f(x),\ f\left(y+\eta(z,x,y)\right) \leq f(z) \ ,\ f\left(x+\eta(y,z,x)\right) \leq f(y)$$

Remark 2.1. Assumption A is just the inequality of the definition of preinvexity with

$$\lambda = 1, \lambda_1 = \lambda_2 = 0$$

Assumption B. Let $\eta: X \times X \times X \to R^n$. Then, for any x, y, z $\in R^n$ for $\lambda \in [0,1]$.

$$\eta(y, y, y + \lambda\eta(x, y, z)) = -\lambda\eta(x, y, z)$$

$$\eta(x, y + \lambda_1\eta, z) = (1 - \lambda_1)\eta(x, y, z)$$

$$\eta(x, y, z + \lambda_2\eta) = (1 - \lambda_2)\eta(x, y, z)$$

Remark 2.2. Every quasimonotone map is an invariant quasimonotone map, but the converse is not necessarily true with

$$\eta(x, y, z) = 2z - x - y$$

$$\eta(z, x, y) = 2y - x - z$$

$$\eta(y, z, x) = 2x - y - z$$

Where $\lambda + \lambda_1 + \lambda_2 = 1$

Example 2.1. Define the map F and η as

$$F(x) = \left(\sin^2 x_1.\cos x_1, \sin^2 x_2.\cos x_2, \sin^2 x_3.\cos x_3\right),\ x \in [0,\pi] \times [0,\pi] \times [0,\pi],$$

$$\eta(x, y, z) = [\frac{\cos y_1(\sin x_1 - \sin y_1)}{\sin z_1}, \frac{\cos y_2(\sin x_2 - \sin y_2)}{\sin z_2}, \frac{\cos y_3(\sin x_3 - \sin y_3)}{\sin z_3}]$$

$$x, y, z \in [0,\pi] \times [0,\pi] \times [0,\pi]$$

$$\eta(y, x, z)^T F(x) = \frac{\cos x_1(\sin y_1 - \sin x_1)}{\sin z_1}.\sin^2 x_1 \cos x_1 +$$

$$\frac{\cos x_2(\sin y_2 - \sin x_2)}{\sin z_2}.\sin^2 x_2 \cos x_2 + \frac{\cos x_3(\sin y_3 - \sin x_3)}{\sin z_3}.\sin^2 x_3 \cos x_3$$

$$= \sum \frac{(\sin y_1 - \sin x_1)\sin^2 x_1.\cos^2 x_1}{\sin z_1} > 0$$

Clearly, F is invariant quasimonotone with respect to η. Let

$$x = (3\pi/4, 3\pi/4, 3\pi/4), y = (\pi/4, \pi/4, \pi/4)$$

Then,

$(y-x)^T F(x) = 3\pi/4\sqrt{2} > 0$, but $(y-x)^T F(y) = -3\pi/4\sqrt{2} < 0$

Thus, F is not quasimonotone.

Definition 2.3. See Ref. 5. Let Γ of R^n be invex set with respect to η. A function f is prequasiinvex with respect to the same η on Γ if, for all x, y, z $\in \Gamma$, $\lambda \in [0,1]$

$f(y) \le f(x)$ implies $f(y + \lambda\eta(x,y,z)) \le f(x)$, z fixed

$f(z) \le f(y)$ implies $f(z + \lambda\eta(x,y,z)) \le f(y)$, x fixed

$f(x) \le f(z)$ implies $f(x + \lambda\eta(x,y,z)) \le f(z)$, y fixed

Lemma 2.1. See Ref .6.Let Γ of R^n be an invex set with respect to η,and let η satisfy Assumption B.Then, a differentiable function f is prequasiinvex with respect to η on Γ if and only if ,for every of points x,y,z $\in \Gamma$,

$f(y) \le f(x)$ Implies $\eta(y,x,z)^T \nabla f(x) \le 0$

Proof. Let Γ of R^n be an invex set with respect to η , let η satisfy Assumption B

Let f is prequasiinvex with respect to η on Γ , we have

$f(y) \le f(x)$ implies $f(y + \lambda\eta(x,y,z)) \le f(x)$

$f(y + \lambda\eta(x,y,z)) - f(x) \le 0$

$f(y + \lambda\eta(x,y,z), -x) = 0$

$\eta(y,x,z)^T \nabla f(x) \le 0$ (Assumption B)

Conversely $f(y) \le f(x)$ implies $\eta(y,x,z)^T \nabla f(x) \le 0$

$f(y + \lambda\eta(x,y,z), -x) = 0$

$f(y + \lambda\eta(x,y,z)) - f(x) \le 0$

$f(y + \lambda\eta(x,y,z)) \le f(x)$

Theorem 2.1. Let Γ of R^n is an invex set with respect to η, and let f be a differentiable function on Γ.If f and η satisfy Assumption B, then f is prequasiinvex with respect to the same η on Γ if and only if ∇f is invariant quasimonotone with respect to the same η on Γ and, for all x, y, z $\in \Gamma$,

$$f(y) \le f(x) \text{ implies. } f\left(y + \lambda\eta(x,y,z)\right) \le f(x)$$

Proof: Suppose that f is prequasiinvex w.r.t η . It is obvious that Inequality (C) is true. Let x, y, z $\in$ Γ be such that

$$\eta(y,x,z)^T \nabla f(x) > 0 \tag{1}$$

Then we have $f(y) > f(x)$.

By lemma 2.1 $f(y) \le f(x)$ implies that $\eta(x,y,z)^T \nabla f(y) \le 0$.

This shows that ∇f is invariant quasimonotone with respect to the same η.

Conversely, suppose that ∇f is invariant quasimonotone with respect to η.

Assume that f is not prequasiinvex with respect to the same η. Then, there exist x, y, z $\in \Gamma$ such that

$$f(y) \le f(x);$$

Furthermore, there exist a $\bar{\lambda} \in (0,1)$ such that

$$f\left(y + \bar{\lambda}\eta(x,y,z)\right) > f(x) \ge f(y). \tag{2}$$

By mean value theorem, there exist $\lambda_1, \lambda_2 \in (0,1)$ such that

$$f\left(y + \bar{\lambda}\eta(x,y,z)\right) - f\left(y + \eta(x,y,z)\right)$$

$$= \left(\bar{\lambda} - 1\right)\eta(x,y,z)^T \nabla f\left(y + \lambda_1\eta(x,y,z)\right) \tag{3}$$

$$f\left(y + \bar{\lambda}\eta(x,y,z)\right) - f(y) = \bar{\lambda}\eta(x,y,z)^T \nabla f\left(y + \lambda_2\eta(x,y,z)\right) \tag{4}$$

$$0 < \lambda_2 < \bar{\lambda} < \lambda_1 < 1 \tag{5}$$

Then, from (2)-(5) and Inequality (C), we have

$$\eta(x,y,z)^T \nabla f\left(y + \lambda_1\eta(x,y,z)\right) < 0 \tag{6}$$

$$\eta(x,y,z)^T \nabla f\left(y + \lambda_2\eta(x,y,z)\right) > 0 \tag{7}$$

From Assumption B, we have

$$\eta\left(y + \lambda_2\eta(x,y,z), y + \lambda_1\eta(x,y,z)\right)$$

$= \ \eta\big(y+\lambda_2\eta(x,y,z), y+\lambda_2\eta(x,y,z)+(\lambda_1-\lambda_2)\eta(x,y,z)\big)$

$= \ \eta\Big(y+\lambda_2\eta(x,y,z), y+\lambda_2\eta(x,y,z)+[(\lambda_1-\lambda_2)/(1-\lambda_2)]\eta\big(x, y+\lambda_2\eta(x,y,z)\big)\Big)$

$= \ -[(\lambda_1-\lambda_2)/(1-\lambda_2)]\eta\big(x, y+\lambda_2(x,y,z)\big)$

$$= \ (\lambda_2-\lambda_1)\eta(x,y,z) \tag{8}$$

$\eta\big(y+\lambda_1\eta(x,y,z), y+\lambda_2\eta(x,y,z)\big)$

$= \ \eta\big(y+\lambda_1\eta(x,y,z), y+\lambda_1\eta(x,y,z)-(\lambda_1-\lambda_2)\eta(x,y,z)\big)$

$= \ \eta\Big(y+\lambda_1\eta(x,y,z), y+\lambda_1\eta(x,y,z)+\eta\big(y, y+(\lambda_1-\lambda_2)\eta(x,y,z)\big)\Big)$

$= \ -\eta\big(y, y+(\lambda_1-\lambda_2)\eta(x,y,z)\big)$

$$= \ (\lambda_1-\lambda_2)\eta(x,y,z) \tag{9}$$

Then, by (6) – (9), it follows that

$\eta\big(y+\lambda_2\eta(x,y,z), y+\lambda_1\eta(x,y,z)^T\nabla f\big(y+\lambda_1\eta(x,y,z)\big)\big)>0$

$\eta\big(y+\lambda_1\eta(x,y,z), y+\lambda_2\eta(x,y,z)^T\nabla f\big(y+\lambda_2\eta(x,y,z)\big)\big)>0$

These two inequalities contradict the invariant quasimonotonicity of ∇f .

3. Invariant Pseudomonotone Maps.

Definition 3.1. Let $\Gamma \subset R^n$. F: $\Gamma \to R^n$ is said to be pseudomonotone on Γ if, for every pair of distinct points x, y, z $\in \Gamma$,

$(y-x)^T f(x) \geq 0$ implies $(y-x)^T f(y) \geq 0$

$(z-y)^T f(y) \geq 0$ implies $(z-y)^T f(z) \geq 0$

$(x-z)^T f(z) \geq 0$ implies $(x-z)^T f(x) \geq 0$.

Definition 3.2. Let Γ of R^n be an invex set with respect to η.Then F: $\Gamma \to R^n$ is said to be invariant pseudomonotone with respect to η on Γ of R^n if, for every pair of distinct points x, y, z $\in \Gamma$

$\eta(z,y,x)^T F(x) \geq 0$ implies $\eta(z,x,y)^T F(y) \geq 0$, z fixed

$\eta(y,x,z)^T F(z) \geq 0$ implies $\eta(y,z,x)^T F(x) \geq 0$, y fixed

$\eta(x,z,y)^T F(y) \ge 0$ implies $\eta(x,y,z)^T F(z) \ge 0$, x fixed

Definition 3.3. A differentiable function f on a subset Γ of R^n is pseduoinvex with respect to η on Γ if, for every pair of distinct points x, y, z $\in \Gamma$

$\eta(z,y,x)^T \nabla f(x) \ge 0$ implies $f(y) \ge f(x)$, z fixed

$\eta(y,x,z)^T \nabla f(z) \ge 0$ implies $f(x) \ge f(z)$, y fixed

$\eta(x,z,y)^T \nabla f(y) \ge 0$ implies $f(z) \ge f(y)$, x fixed

Remark 3.1. Every pseudomonotone map is an invariant pseudomonotone map with

$\eta(x,y,z) = 2z - x - y$, but the converse is not necessarily true.

Example 3.1 Define the map F and η as

$$F(x_1,x_2,x_3) = (1,\cos x_2,\cos x_3),\ (x_1,x_2,x_3) \in (\pi/2,\pi/2,\pi/2)$$

$$\eta(x,y,z) = [\sin x_1 - \sin y_1, (\sin x_2 - \sin y_2)/\cos y_2, (\sin x_3 - \sin y_3)/\cos y_3],$$

$$x = (x_1,x_2,x_3), y = (y_1,y_2,y_3), z = (z_1,z_2,z_3) \in (\pi/2,\pi/2,\pi/2)$$

Clearly, F is invariant pseudomonotone with respect to η .Let

$x = (\pi/3,0,0), y = (\pi/6,\pi/6,\pi/6), z = (\pi/4,\pi/4,\pi/4)$

Then,

$$(y-x)^T F(x) = 0 \text{ and } (y-x)^T F(y) = (\pi/6)\left(\frac{1}{\sqrt{2}} - 1\right) < 0$$

Thus, F is not pseudomonotone.

Remark 3.2. Every invariant monotone map is an invariant pseudomonotone map with respect to the same η, but the converse is not necessarily true.

Example 3.2. Define the maps F and η as

$$F(x) = \cos^2 x,\ x \in (-\pi/2,\pi/2,\pi/2)$$

$$\eta(x,y,z) = \cos y + \cos z - 2\cos x,\ x,y,z \in (-\pi/2,\pi/2,\pi/2)$$

Clearly, F is invariant pseudomonotone with respect to η on

$\left(-\pi/2, \pi/2, \pi/2\right)$.

Let

$x = -\pi/6$, $y = \pi/4$, z = $\pi/3$

Then,

$\eta(z, x, y)^T F(x) + \eta(z, y, x)^T F(y) > 0$

Thus, F is not invariant monotone with respect to η on $\left(-\pi/2, \pi/2, \pi/2\right)$.

Remark 3.3. Every invariant pseudomonotone map is an invariant quasimonotone map with respect to the same η but the converse is not true.

Example 3.3. Define the maps F and η as

F(x) $= \sin^2 x.\cos x$, $x \in [0, \pi]$,

$\eta(x, y, z) = \cos y(\sin x - \sin y)/\sin z \quad x, y, z \in [0, \pi]$.

Clearly, F is invariant quasimonotone with respect to η .Let

$x = \pi/2$, $y = \pi/4$, $z = \pi/6$.

Then,

$\eta(z, y, x)^T F(x) = 0$, but $\eta(z, x, y)^T F(y) > 0$.

Thus, F is not invariant pseudomonotone with respect to η .

It is well known that every pseudoconvex function is quasiconvex. This result can be generalized to the invex-type function. The details are given in the following lemma.

Lemma 3.1. Let f and η satisfy Assumption B.Assume that the differentiable function f is pseudoinvex with respect to η on an invex set Γ of R^n and that, for all x, y, z $\in \Gamma$,

(c) $f(y) \le f(x)$ implies $f\left(y + \eta(x, y, z)\right) \le f(x)$.

Then, f is prequasiinvex with respect to the same η on Γ.

Proof. Suppose f is pseudoinvex with respect to η on Γ. Assume that f is not prequasiinvex with respect to η . Then there exist x, y, z $\in \Gamma$ such that

$f(y) \le f(x)$; z fixed

Furthermore, there exist a $\bar{\lambda} \in (0,1)$ such that for $\bar{x} = y + \bar{\lambda}\eta(x,y,z)$,

$f(\bar{x}) > f(y) \ge f(x)$.

From Inequality (c) and the above inequalities, there exists $\bar{y} = y + \lambda^*\eta(x,y,z)$, for $\lambda^* \in (0,1)$, such that

$$f(\bar{y}) = \max_{\lambda\in[0,1]} f\left(y + \lambda\eta(x,y,z)\right).$$

Then, it follows that

$$\eta(x,y,z)^T \nabla f(\bar{y}) = 0.$$

From Assumption B, we have

$$\eta(x,\bar{y},z) = (1-\lambda^*)\eta(x,y,z), \quad \eta(y,\bar{y},z) = -\lambda^*\eta(x,y,z),$$

Hence,

$$\eta(x,\bar{y},z)\nabla f(\bar{y}) = (1-\lambda^*)\eta(x,y,z)^T \nabla f(\bar{y}) = 0.$$

Since f is pseudoinvex with respect to η, it holds that

$$f(\bar{y}) \le f(x),$$

Which is a contradiction. Thus, f is prequasiinvex with respect to η.

Theorem 3.1. Let Γ of R^n be an open invex set with respect to η, let f be differentiable on Γ of R^n, and let f and η satisfy Assumption A and B respectively.

Then, f is pseudoinvex with respect to η on Γ if and only if ∇f is invariant pseudomonotone with respect to η on Γ.

Proof. Suppose that f is pseudoinvex with respect to η on Γ. Let x, y,z $\in \Gamma$, $x \ne y \ne z$

be such that $\eta(x,y,z)^T \nabla f(y) \ge 0$.

We need to prove that

$$\eta(y,x,z)^T \nabla f(x) \le 0.$$

Assume the contrary, i.e.,

$$\eta(y,x,z)^T \nabla f(x) > 0. \tag{1}$$

By the pseudoinvexity of f with respect to η, we have x, y, z $\in \Gamma$

$$\eta(y,x,z)^T \nabla f(x) \geq 0 \text{ implies } f(y) \geq f(x) \tag{2}$$

From Lemma 3.1, every pseudoinvex function is also prequasiinvex with respect to the same η.It follows from (2) and lemma 3.1 that

$$\eta(y,x,z)^T \nabla f(x) \leq 0.$$

Which contradicts (1).Therefore, ∇f is invariant pseudomonotone with respect to η.

Conversely, suppose that ∇f is invariant pseudomonotone on Γ.

Let x, y, z $\in \Gamma$, $x \neq y \neq z$,be such that $\eta(x,y,z)^T \nabla f(y) \geq 0$. (3)

We need to prove that $f(x) \geq f(y)$.

Assume the contrary i.e.,

$$f(x) < f(y) \tag{4}$$

By the mean- value theorem, we have

$$f\left(y+\eta(x,y,z)\right) - f(y) = \eta(x,y,z)^T \nabla f\left(y+\bar{\lambda}\eta(x,y,z)\right) \tag{5}$$

for some $\bar{\lambda} \in (0,1)$. By assumption A and B it follows that

$$f\left(y+\eta(x,y,z)\right) \leq f(x) \tag{6}$$

$$\eta\left(y,y,y+\bar{\lambda}\eta(x,y,z)\right) = -\bar{\lambda}\eta(x,y,z) \tag{7}$$

Now, from (4) - (7), we have

$$\eta\left(y,y,y+\bar{\lambda}\eta(x,y,z)\right)^T \nabla f\left(y+\bar{\lambda}\eta(x,y,z)\right) > 0 \tag{8}$$

Since ∇f is invariant psedomonotone with respect to η, it follows from (8) that

$$\eta\left(y+\bar{\lambda}\eta(x,y.z),y,y\right)^T \nabla f < 0.$$

From Assumption B, we have

$\eta(x,y,z)^T \nabla f(y) < 0$.

Which contradicts (3). Hence, f is pseudoinvex with respect to η.

4. Strictly Invariant Pseudomonotone Maps

Definition 4.1. A map F is strictly pseudomonotone on a set Γ of R^n if, for every pair of distinct points x, y, z $\in \Gamma$,

$(y-x)^T F(x) \geq 0$ implies $(y-x)^T F(y) > 0$

$(z-y)^T F(y) \geq 0$ implies $(z-y)^T F(z) > 0$

$(x-z)^T F(z) \geq 0$ implies $(x-z)^T F(x) > 0$.

Definition 4.2. Let Γ of R^n be an invex set with respect to η. A map F is strictly invariant Pseudomonotone with respect to η on Γ if, for every pair of distinct points

x, y, z $\in \Gamma$,

$\eta(\mathrm{z,y,x})^T F(x) \geq 0$ implies $\eta(\mathrm{z,x,y})^T F(y) < 0$

$\eta(\mathrm{y,x,z})^T F(z) \geq 0$ implies $\eta(\mathrm{y,z,x})^T F(x) < 0$

$\eta(\mathrm{x,z,y})^T F(y) \geq 0$ implies $\eta(\mathrm{x,y,z})^T F(z) < 0$.

Remark 4.1. Every strictly pseudomonotone map is a strictly invariant pseudomonotone map with $\eta(x,y,z) = 2z - x - y$, but the converse is not necessarily true.

Example 4.1. Define the maps F and η as

F(x) = $\sin x + \cos x, \quad x \in (0,\pi)$

$\eta(x,y,z) = (\sin y + \cos y)(\cos x - \cos y)(\sin z + \sin y) \quad x,y,z \in (0,\pi)$

Clearly, F is strictly invariant pseudomonotone with respect to η on $(0,\pi)$.

Let,

$x = 3\pi/4$, $y = \pi/4$, z $= y = \pi/3$.

Then, F is not strictly pseudomonotone on $(0,\pi)$.

Remark 4.2. Every strictly invariant monotone map is a strictly invariant psedomonotone map with respect to the same η map, but the converse is not necessarily true.

Example 4.3. Define the maps F and η as

$$F(x) = \sin x.\cos^2 x, \quad x \in (-\pi/2, \pi/2, \pi/2)$$

$$\eta(x, y, z) = \sin y \cos z(\cos y - \cos x) \quad x, y, z \in (-\pi/2, \pi/2, \pi/2) \ .$$

Clearly, F is invariant pesudomonotone with respect to η on $(-\pi/2, \pi/2, \pi/2)$.

Let

$x = -\pi/6$, $y = \pi/6$, $z = \pi/6$.

Then,

$\eta(y, x, z)^T F(x) = 0$ and $\eta(x, y, z)^T F(y) = 0$.

Thus, F is neither strictly invariant psedomonotone nor strictly invariant monotone with respect to the same on $(-\pi/2, \pi/2, \pi/2)$.

Definition 4.3. Let Γ of R^n be an open invex set with respect to η.A differentiable function f on Γ is strictly pseudoinvex with respect to η on Γ if, for every pair of distinct points x,y,z $\in \Gamma$,

$\eta(y, x, z)^T \nabla f(x) \geq 0$ implies f(y)>f(x).

Theorem 4.1. Let Γ of R^n be an open invex set with respect to η, and let f be differentiable on Γ.If f and η satisfy Assumption A and C respectively, then f is strictly pseudoinvex with respect to η on Γ if and only if ∇f is strictly invariant pseudomonotone with respect to η on Γ.

Proof. Suppose that f is strictly pseudoinvex with respect to η on Γ.

Let x, y, z $\in \Gamma$, $x \neq y \neq z$, such that

$$\eta(y, x, z)^T \nabla f(x) \geq 0. \tag{1}$$

We need to show that

$\eta(x, y, z)^T \nabla f(y) < 0.$

On the contrary, we assume that

$$\eta(x,y,z)^T \nabla f(y) \geq 0.$$

From the strict pseudoinvexity of f with respect to η , it follows that

$$f(x) > f(y). \tag{2}$$

On the other hand, from the strict pseudoinvexity of f with respect to η, (1) implies that

$$f(y) > f(x),$$

which contradicts (2).

Conversely, suppose that ∇f is strictly pseudoinvex with respect to η on C.

Let x, y, z $\in \Gamma$, $x \neq y \neq z$, be such that

$$\eta(y,x,z)^T \nabla f(x) \geq 0 \tag{3}$$

We need to show that

$$f(y) > f(x) \tag{4}$$

On the contrary, we assume that

$$f(y) \leq f(x) \tag{5}$$

By mean value theorem, we assume that

$$f\left(x + \eta(y,x,z)\right) - f(x) = \eta(y,x,z)^T \nabla f\left(x + \bar{\lambda}\eta(y,x,z)\right) \tag{6}$$

For some $0 < \bar{\lambda} < 1$. By Assumption A,

$$f(x + \eta(y,x,z)) \leq f(y). \tag{7}$$

Now, from (4)-(7) and Assumption B, we have

$$\eta\left(x, x + \bar{\lambda}\eta(y,x,z), z\right)^T \nabla f\left(x + \bar{\lambda}\eta(y,x,z)\right)$$

$$= -\bar{\lambda}\eta(y,x,z)^T \nabla f\left(x + \bar{\lambda}\eta(y,x,z)\right) \geq 0 \tag{8}$$

Since ∇f is strictly invariant pseudomonotone with respect to η,
We conclude that

$$\eta\left(x+\bar{\lambda}\eta(y,x,z),x,z\right)^T \nabla f\left(x\right)<0. \tag{9}$$

Again, from Assumption B, we note that

$$\eta\left(x+\bar{\lambda}\eta(y,x,z),x,z\right)^T$$

$$= \eta\left(x+\bar{\lambda}\eta(y,x,z),x+\bar{\lambda}\eta(y,x,z)+\eta\left(x,x+\bar{\lambda}\eta(y,x,z),z\right)\right)$$

$$= -\eta\left(x,x+\bar{\lambda}\eta(y,x,z),z\right)$$

$$= \bar{\lambda}\eta(y,x,z).$$

Thus, it follow from (9) that

$$\eta(y,x,z)^T \nabla f(x)<0,$$

which contradicts (3)

Hence, $f(y)>f(x)$.

5. Conclusion

In this paper, we have introduced concepts of generalized invariant Quasi and pseudo Monotonicities and established their relations with generalized invexities.

References

[1] HANSON, M.A., On Sufficiency of the Kuhn Tucker Conditions, Journal of Mathematical Analysis and Applications, Vol. 80, pp. 545-550, 1981.

[2] WEIR, T., and MOND, B., Preinvex Functions in Multiple-Objective Optimization Journal of Mathematical Analysis and Applications, Vol 136, pp. 29-38, 1988.

[3] WEIR, T., and JEYAKUMAR, V., A Class of Nonconvex Functions and Mathemati- cal Programming, Bulletin of the Australian Mathematical Society, Vol. 38, pp. 177- 189, 1988.

[4] MOHAN, S.R., and NEOGY, S.K., On Invex Sets and Preinvex Functions, Journal of Mathematical Analysis and Applications, 189, pp. 901-908, 1995.

[5] KARAMARDIAN, S., and SCHAIBLE, S., Seven Kinds of Monotone Maps, Journal Of Optimization Theory and Applications, Vol. 66, pp.37-46, 1990.

[6] PINI, R., and SINGH, C., Generalized Convexity and Generalized Monotonicity, Journal of Information and Optimization Sciences, Vol. 20, pp. 215-233, 1999.
[7] RUIZ-GARZON, G., OSUNA-GOMEZ, R., and RUFIAN-LIZANA, A., Generalized Invex Monotonicity, European Journal of Operational Research, Vol.144, Pp.501-502, 2003.
[8] X.M.Yang, X.Q.Yang and K.L.Teo, Generalized invexity and generalized invariant monotonicity, journal of optimization theory and application, Vol.177, No 3, Pp.607-625, 2003.