Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 5, Number 2 (2013), pp. 87-92

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Pathos semirelib graph of a tree

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Abstract

In this communucations, the concept of pathos semirelib graph of a tree is introduced. We present a characterization of those graphs whose pathos semirelib graphs are planar, outer planar, eulerian, hamiltonian with crossing number one.

AMS subject classification: 10C75, 10C10.
Keywords: Blocks, edge degree, inner vertex number, line graph, path number, regions.

1. Introduction

The concept of block edge cut vertex graph was introduced by Venkanagouda M Goudar [5]. For the graph G(p,q), if $B = u_1, u_2, \ldots, u_r : r \geq 2$ is a block of G, then we say that the vertex u_i and the block B are incident with each other. If two blocks B_1 and B_2 are incident with a common cutvertex, then they are adjacent blocks.

All undefined terminology will conform with that in Harary [2]. All graphs considered here are finite, undirected, planar and without loops or multiple edges.

The pathos semirelib graph of a tree T denoted by $P_s(T)$ is the graph whose vertex set is the union of set of edges, set of blocks, set of regions and the path of pathos of T in which two vertices are adjacent if and only if the corresponding edges of T are adjacent, the corresponding edges lies on the blocks, the corresponding edges lies on the region and the edges lies on the pathos.

The edge degree of an edge uv is the sum of the degree of the vertices of u and v. For the planar graph G, the inner vertex number i(G) of a graph G is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. A graph G is said to be minimally nonouterplanar if $i(G) = 1$ as was given by Kulli [4].

2. Preliminary Notes

We need the following results to prove further results.

Theorem 2.1. [2] If G is a (p,q) graph whose vertices have degree d_i then the line graph L(G) has q vertices and q_L edges, where $q_L = -q + \frac{1}{2}\sum d_i^2$ edges.

Theorem 2.2. [2] The line graph L(G) of a graph is planar if and only if G is planar, $\Delta(G) \leq 4$ and if $degv = 4$, for a vertex v of G, then v is a cutvertex.

Theorem 2.3. [3] A graph is planar if and only if it has no subgraph homeomorphic to K_5 or $K_{3,3}$.

Theorem 2.4. [4] A graph is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.

3. Main Results

We start with few preliminary results.

Lemma 3.1. For any tree T, $L(T) \subseteq P_s(T)$.

Lemma 3.2. If a tree T contains a block as K_2, then it becomes a pendent vertex in $P_s(T)$.

In the following theorem we obtain the number of vertices and edges of a pathos semirelib graph of a tree.

Theorem 3.3. For any nontrivial tree T, the pathos semirelib graph $P_s(T)$ whose vertices have degree d_i and path number k has $(q+r+b+k)$ vertices and $\frac{1}{2}\sum d_i^2 + \sum q_j + q$ edges where r and b be the number of regions and blocks respectively.

Proof. By the definition of $P_s(T)$, the number of vertices is the union of edges, regions, blocks and the path of pathos of T. Hence $P_s(T)$ has $(q + r + b + k)$ vertices. Further by the Theorem 2.1, number of edges in L(T) is $q_L = -q + \frac{1}{2}\sum d_i^2$. Thus the number of edges in $P_s(T)$ is the sum of the number of edges in L(T), the number of edges bounded by the regions which is q, the number of edges lies on the blocks is $\sum q_j$

and the number of edges lies on the path of pathos which is q. Hence $E[P_s(G)] = -q + \frac{1}{2}\sum d_i^2 + q + \sum q_j + q = \frac{1}{2}\sum d_i^2 + \sum q_j + q.$ ■

Theorem 3.4. If a tree T is a path P_2, then $P_s(T) = K_{1,3}$.

Proof. Result is obvious. ■

Theorem 3.5. For any edge in a tree T with edge degree n, the degree of the corresponding vertex in $P_s(T)$ is $n + 1$.

Proof. Suppose an edge $e \in E(T)$ have degree n. By the definition of pathos semirelib graph, the corresponding vertex in $P_s(T)$ has $n - 1$. Since edge is a block, we have the degree of the vertex is $n - 1 + 1 = n$. Further, the edge lies on a path of pathos. Clearly degree of e_i is $n + 1$. ■

Theorem 3.6. If T is a tree with n edges then $P_s(T)$ contains n pendent vertices.

Proof. Let $e_1, e_2, \ldots, e_n \in E(T)$, $b_1 = e_1, b_2 = e_2, \ldots, b_n = e_n$ be the blocks and $r_1, r_2, \ldots, r_k$ be the regions of G. By the definition of line graph L(T), $e_1, e_2, \ldots, e_n$ form a subgraph without isolated vertex. Consider that T does not contain any pendent pathos. By the definition of $P_s(T)$, the region vertices are adjacent to these vertices to form a graph without isolated vertex. Since there are n blocks which are K_2, we have each $b_1 = e_1, b_2 = e_2, \ldots, b_n = e_n$ are adjacent to $e_1, e_2, \ldots, e_n$. Hence $P_s(T)$ contains n pendent vertices. If a tree T contains a pendent pathos, then the vertex $b_i = e_i$ is adjacent to the pathos vertex Pi to form a pendent vertex. Hence the result follows. ■

Theorem 3.7. For any tree T, $P_s(T)$ is separable.

Proof. $e_1, e_2, \ldots, e_n \in E(T)$, $r_1, r_2, \ldots, r_k$ be the regions and $b_1, b_2, \ldots, b_m$ be the blocks which are K_2. By the definition of line graph, $e_1, e_2, \ldots, e_n$ form a subgraph without isolated vertex. Since each block is K_2, clearly each block of T contains exactly one edge and each region of G has at least three edges. Further, the pendent pathos of T becomes a pendent vertex. Hence by the definition of $P_s(T)$ is separable.

In the following theorem we obtain the condition for the planarity on pathos semirelib graph of a tree. ■

Theorem 3.8. For any non trivial tree T, the $P_s(T)$ is planar if and only if T is a tree such that $\Delta(T) \leq 3$.

Proof. Suppose $P_s(T)$ is planar. Assume that $\exists v_i \in G$ such that $degv_i \geq 4$. Suppose $degv_i = 4$ and e_1, e_2, e_3, e_4 are the edges incident to v_i. By the definition of line graph, e_1, e_2, e_3, e_4 form K_4 as an induced subgraph. In $P_s(T)$, the region vertex r_i is adjacent with all vertices of L(T) to form K_5 as an induced subgraph. Further the corresponding block vertices $b_1, b_2, b_3, \ldots, b_{n-1}$ of of blocks $B_1, B_2, B_3, \ldots, B_n$ in T are adjacent to vertices of K_4 and the pathos vertices are adjacent to the edges lies on the paths of

T, which forms graph homeomorphic to K_5. By the Theorem 2.3, it is non planar, a contradiction.

Conversely, Suppose $degv \leq 3$ and let e_1, e_2, e_3 be the edges of T incident to v. By the definition of line graph e_1, e_2, e_3 form K_3 as a subgraph. By the definition of $P_s(T)$, the region vertex r_i is adjacent to e_1, e_2, e_3 to form K_4 as a subgraph. Further, by the lemma 2.2, the blocks $b_1, b_2, b_3, \ldots, b_n$ of T with n vertices such that $b_1 = e_1, b_2 = e_2, \ldots, b_{n-1} = e_{n-1}$ becomes $p-1$ pendant vertices. The pathos vertices have degree at most three. Hence $P_s(T)$ is planar.

In the following theorem we obtain the condition for the outer planarity on pathos semirelib graph of a tree. ■

Theorem 3.9. For any non trivial tree T, $P_s(T)$ is outer planar if and only if T is a path $P_n, n \leq 3$.

Proof. Suppose $P_s(T)$ is outer planar. We have the following cases.

Case 1. Assume that T is a tree with at least one vertex v such that $degv = 3$. Let e_1, e_2, e_3 be the edges of T incident to v. By the definition of line graph e_1, e_2, e_3 form K_3 as a subgraph.In $P_s(T)$, the region vertex r_i is adjacent to e_1, e_2, e_3 to form K_4 as induced subgraph. Further by the lemma 3.2, $b_1 = e_1, b_2 = e_2, \ldots, b_{n-1} = e_{n-1}$ becomes n-1 pendant vertices in $P_s(T)$. Clearly $i[P_s(T) \geq 1]$, which is non-outer planar, a contradiction.

Case 2. Assume that T is path $P_n, n \geq 4$. By the definition of line graph, $L[P_4] = P_3$. Since the edges e_1, e_2 lies on the pathos and the region,clearly these vertices of $P_s(T)$ form $K_4 - x$. Similarly, the edge e_3 lies on the same pathos and the same region. Clearly the vertices e_1, e_2, e_3, r_1, p_1 of $P_s(T)$ form a graph such that the vertex e_2 becomes the inner vertex, which is a contradiction. Hence the result the follows.

Conversely, suppose T is a path $P_n, n \leq 3$. Let $e_1, e_2, e_3 \in E(T)$. By the definition of line graph $L[P_3] = P_2$. Further by the lemma 3.2, $b_1 = e_1, b_2 = e_2, b_3 = e_3$ becomes three pendant vertices and it becomes a caterpillar. Further the region vertex r_1 is adjacent to all the vertices of $L[P_3]$. Also the pathos vertex p_1 is adjacent to the vertices e_1, e_2, e_3 of P_s, which is is outer planar. Hence the result follows. ■

In the following theorem we obtain the condition for the minimally non outer planar on pathos semirelib graph of a tree.

Theorem 3.10. For any tree T, $P_s(T)$ is minimally non-outer planar if and only if G is $K_{1,3}$.

Proof. Suppose $P_s(T)$ is minimally non-outer planar. Assume that $T \neq K_{1,3}$. Consider the following cases.

Case 1. Assume that $T = K_{1,n}$, for $n \geq 4$ then there exist at least one vertex of degree at least 4. Suppose $degv = 4$ for any $v \in T$. By the definition of line graph, $L[K_{1,4}] = K_4$.

By the definition of $P_s(T)$, it is minimally non-outer planar, a contradiction.

Case 2. Suppose $T \neq K_{1,n}$. By the Theorem 3.9, $P_s(T)$ is non-outer planar, a contradiction.

Conversely, suppose $T = K_{1,3}$ and let $e_1, e_2, e_3 \in E(T)$. By the definition of line graph, $L[K_{1,3}] = K_3$. Let r_1 be the region vertex and p_1, p_2 be the pathos vertices in $P_s(G)$ such that r_1 is adjacent to all vertices of K_3 to form K_4. Clearly $i[K_4] = 1$. Hence G is minimally non-outer planar. ■

In the following theorem we obtain the condition for the non eulerian on pathos semirelib graph of a tree.

Theorem 3.11. For any tree T, $P_s(G)$ is always non Eulerian.

Proof. Suppose T is a tree. In a tree each edge is a block and hence $b_1 = e_1, b_2 = e_2, \ldots, b_{n-1} = e_{n-1} \forall e_{n-1} \in E(T)$ and $\forall b_{n-1} \in V[P_s(T)]$. In $P_s(T)$, the block vertex b_i is always a pendent vertex, which is non Eulerian. ■

In the following theorem we obtain the condition for the hamiltonian on pathos semirelib graph of a tree.

Theorem 3.12. For any tree T, $P_s(T)$ is non hamiltonian.

Proof. Suppose T is a tree. In a tree each edge is a block and hence $b_1 = e_1, b_2 = e_2, \ldots, b_{n-1} = e_{n-1} \forall e_{n-1} \in E(T)$ and $\forall b_{n-1} \in V[P_s(T)]$. In $P_s(T)$, the block vertex b_i is always a pendent vertex, which is non hamiltonian. ■

In the following theorem we obtain the condition for the crossing number one on pathos semirelib graph of a tree.

Theorem 3.13. For any tree T, $P_s(T)$ has crossing number one if and only if T is a tree with $degv \leq 3, \forall v \in (T)$ and an unique vertex of degree 4.

Proof. Suppose $P_s(T)$ has crossing number one. Assume that T is a tree with at least two vertices v_i, v_j of degree four. Let $e_{i1}, e_{i2}, e_{i3}e_{i4} \in E(T)$ be the edges incident to v_i and $e_{j1}, e_{j2}, e_{ij3}e_{j4} \in E(T)$ be the edges incident to v_j. By the definition of line graph, each star $v_i andv_j$ form two induced subgraphs as K_4. By the definition of $P_s(G)$, the region vertex r, pathos vertices and the block vertex b are adjacent to $e_{i1}, e_{i2}, e_{i3}, e_{i4}, e_{j1}, e_{j2}, e_{j3}e_{j4}$ to form two K_5 subgraphs. This gives $C[P_s(T) \geq 2$, a contradiction.

Conversely, suppose T is a tree with $deg(v) \leq 3, \forall v \in (G)$ and an unique vertex of degree 4. Let v_i be the unique vertex of degree 4 and $e_{i1}, e_{i2}, e_{i3}e_{i4} \in E(T)$ be the edges incident to v_i. By the definition of line graph, $e_{i1}, e_{i2}, e_{i3}e_{i4}$ forms K_4 as induced subgraph. Let $b_{i1} = e_{i1}, b_{i2} = e_{i2}, b_{i3} = e_{i3}$ and $b_{i4} = e_{i4}$ be the blocks of T. By the definition of $P_s(T)$, b_{ij} is adjacent to $e_{ij} \forall j = 1, 2, 3, 4$. Further the region vertex r_i and the pathos vertices are adjacent to all $e_{i1}, e_{i2}, e_{i3}, e_{i4}$ to form K_5 as induced subgraph. Clearly $C[P_s(T)] = 1$. ■

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