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Existence and Uniqueness of Solutions to Partial Functional Differential Equations in Banach Spaces

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Abstract

This paper examines the existence and uniqueness of mild solutions to a class of partial functional differential equations by using the the theory of analytic semigroups. The main technique in this paper is the use of fractional powers of operators combined with the Banach contraction mapping principle.

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1. Introduction

In this paper, we study the existence and uniqueness of mild solutions to a class of partial functional differential equation

$$\begin{aligned} \frac{d}{dt}\left[u(t) + f(t, u(t))\right] &= Au(t) + g(t, u(t)), \quad t_0 < t < t_1 \\ u(t_0) &= x_0 \end{aligned} \tag{1}$$

where A is the infinitesimal generator of an analytic C_0 semigroup $T(t)$ on a Banach space $(X, \|\cdot\|)$ [9], [12], [13] satisfying the exponential stability, i.e.,

$$T(t) \leq Me^{\omega t}, \quad \text{for } \omega \geq 0, M \geq 1, 0 \leq t < \infty, \tag{2}$$

f, g are continuous functions in X. We assume that $f(t, x), g(t, x)$ are locally Hölder continuous in t and locally Lipschitz continuous in x. That is, for each point (t, x) of U, there exist a neighbourhood $V \subset U$, constants $L = L(t, x, V) > 0$ and $0 < \gamma \leq 1$ such that

$$\|f(s_1, y_1) - f(s_2, y_2)\| \leq L\left(|s_1 - s_2|^\gamma + \|y_1 - y_2\|_\alpha\right),$$

for $(s_1, y_1) \in V, (s_2, y_2) \in V$, where X is a real or complex Banach space with norm $\|\cdot\|$, A^α is a closed linear, invertible operator with domain $D(A^\alpha)$ endowed with the graph norm $\|\cdot\|_\alpha$ of A^α with

$$\|x\|_\alpha = \left(\|x\|^2 + \left\|A^\alpha x\right\|^2\right)^{1/2}, \qquad x \in D(A^\alpha).$$

Henry [10], Taira [16] proved the existence and uniqueness of solutions to a semilinear equation

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + f(t, u(t)), \quad 0 < t < \mathbf{T} \\ u(0) &= u_0 \end{aligned} \tag{3}$$

with $f : [0, \mathbf{T}) \longrightarrow X$ Lipschitz and Hölder continuous and A generates C_0 semigroups on a Banach space X satisfying (2). See also [1], [3], [4], [9] [13] for more on (3).

Existence and uniqueness of pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to (1) was proved in [18] where A is the infinitesimal generator of analytic semigroups satisfying (2) and f, g are continuous functions.

Also Diagana [6] investigated the existence and uniqueness of pseudo almost periodic mild solutions to (1) with f, g Lipschitz and Hölder continuous and A generates C_0 semigroups satisfying (2).

The existence and uniqueness of pseudo almost automorphic mild solutions to (1) was proved in [7] where A was replaced with $A(t)$ under the same assumptions as above.

In this paper, we show the existence and uniqueness of mild solutions to (1) and obtain our results by using fractional powers of linear operators and the Banach contraction mapping principle (Banach fixed-point theorem).

The rest of this paper is organized as follows: In section 2 we recall some basic defnitions, lemmas and preliminary facts which will be needed in the prove of our main results in section 3.

In proving the existence and uniqueness of solution of (1), we show first that the map

$$\Phi : [t_0, t_0 + \tau] \longrightarrow X \qquad \text{defined by}$$

$$\begin{aligned} \Phi(u)(t) = T(t - t_0)\left[x_0 + f(t_0, u(t_0))\right] - f(s, u(s)) - \int_{t_0}^{t} AT(t - s) f(s, u(s)) ds \\ + \int_{t_0}^{t} T(t - s) g(s, u(s)) ds \end{aligned}$$

is a strict contraction.

2. Preliminary results

Let X be a real or complex Banach space and A a densely defined closed linear operator with domain $D(A)$.

2.1. Fractional powers of operators

We define fractional powers of operators of certain unbounded linear operators [9], [10], [13] and we concentrate mainly on fractional powers of operators for which $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$. The results obtained here will be used in section 3.

For $\alpha > 0$ we define the fractional power $A^{-\alpha}$ by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt$$

with $A^{\alpha} = \left(A^{-\alpha}\right)^{-1}$ since $A^{-\alpha}$ is one to one. For $0 < \alpha \leq 1$, A^{α} is a closed and densely defined linear operator (i.e., the domain $D(A^{\alpha}) \supset D(A)$ is dense in X) and $X_{\alpha} =$ the space $D(A^{\alpha})$ endowed with the graph norm $\|\cdot\|_{\alpha}$ of A^{α} with

$$\|x\|_{\alpha} = \left(\|x\|^2 + \left\|A^{\alpha} x\right\|^2\right)^{1/2}, \qquad x \in D(A^{\alpha}).$$

Proposition 2.1.

(i) The space X_{α} is a Banach space.

(ii) The graph norm $\|x\|_{\alpha}$ is equivalent to the norm $\left\|A^{\alpha} x\right\|$.

(iii) If $0 < \alpha < \beta < 1$, then we have $X_{\beta} \subset X_{\alpha}$ with continuous injection.

Proof. See [16]. ∎

Lemma 2.2. Let A be the infinitesimal generator of an analytic semigroup $T(t)$. If $0 \in \rho(A)$ then

(i) for $t > 0$, $\alpha \geq 0$ the operator $A^{\alpha} T(t)$ is bounded and

$$\left\|A^{\alpha} T(t)\right\| \leq C_{\alpha} t^{-\alpha} e^{-\delta t}$$

(ii) for $0 < \alpha \leq 1$ and $x \in D(A^{\alpha})$, we have

$$\|(T(t) - I)x\| \leq C_{\alpha} t^{\alpha} \left\|A^{\alpha} x\right\|.$$

Proof. Using the following estimates:

$$\|T(t)\| \leq C e^{-\delta t}$$
$$\|AT(t)\| \leq C_1 t^{-1} e^{-\delta t} \qquad \text{for } t > 0.$$

So for $m = 1, 2, 3, \ldots$

$$\begin{aligned}\left\|A^m T(t)\right\| = \left\|(AT(t/m))^m\right\| &\le \|(AT(t/m))\|^m \\ &\le (C_1 t^{-1} e^{-\delta t/m})^m = C_m t^{-m} e^{-\delta t}.\end{aligned}$$

If $0 < \alpha < 1$, $t > 0$,

$$\left\|A^\alpha T(t)\right\| = \left\|AT(t) \cdot A^{-(1-\alpha)}\right\|, \qquad A^\alpha = AA^{-(1-\alpha)}.$$

Putting

$$A^{-(1-\alpha)} = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} T(s) ds$$

we have

$$\begin{aligned}\left\|A^\alpha T(t)\right\| &= \left\|AA^{-(1-\alpha)}\right\| \\ &= \left\|\frac{1}{\Gamma(1-\alpha)} \int_0^\infty AT(t) s^{-\alpha} T(s) ds\right\| \\ &\le \frac{1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} \left\|AT(t+s)\right\| ds \\ &\le \frac{C_1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} (t+s)^{-1} e^{-\delta(t+s)} ds \\ &= \frac{C_1}{\Gamma(1-\alpha)} \int_0^\infty s^{-\alpha} t^{-1} \left(1 + \frac{s}{t}\right)^{-1} e^{-\delta t \left(1 + \frac{s}{t}\right)} ds.\end{aligned}$$

Let $u = \frac{s}{t}$, $s = ut$, $ds = tdu$, so

$$\begin{aligned}\left\|A^\alpha T(t)\right\| &\le \frac{C_1}{\Gamma(1-\alpha)} \int_0^\infty (ut)^{-\alpha} t^{-1} (1+u)^{-1} e^{-\delta t(1+u)} t du \\ &\le \frac{C_1 e^{-\delta t}}{\Gamma(1-\alpha) t^\alpha} \int_0^\infty u^{-\alpha} e^{-u} du \\ &= \frac{C_1 e^{-\delta t}}{\Gamma(1-\alpha) t^\alpha} \Gamma(1-\alpha) \\ &= C_\alpha t^{-\alpha} e^{-\delta t}.\end{aligned}$$

Part (ii) follows from

$$(T(t) - I)x = -\int_0^t AT(s) x ds,$$

that is,

$$\frac{d}{dt}(T(t)-I)x = -\frac{d}{dt}\int_0^t AT(s)xds$$

$$\begin{aligned}\|(T(t)-I)x\| &= \left\|\int_0^t A^{1-\alpha}T(s)A^{\alpha}xds\right\| \\ &\leq C\int_0^t s^{\alpha-1}\left\|A^{\alpha}x\right\|ds \\ &= C_{\alpha}t^{\alpha}\left\|A^{\alpha}x\right\|.\end{aligned}$$

■

2.2. Lipschitz and Hölder continuous functions in Banach spaces

Let X be a Banach space and let $A : X \longrightarrow X$ be a densely defined closed linear operator, that is, the domain of definition $D(A)$ of A is dense in the space X.

Definition 2.3. A function $f : [0, \mathbf{T}) \longrightarrow X$ is *Lipschitz continuous* on $[0, \mathbf{T})$ if there is a constant L such that

$$\|f(t) - f(s)\| \leq L\,\|t-s\| \qquad \text{for } 0 \leq s \leq t < \mathbf{T}.$$

It is *locally Lipschitz continuous* if every $t \in [0, \mathbf{T})$ has a neighbourhood in which f is Lipschitz continuous.

Definition 2.4. A function $f : [0, \mathbf{T}) \longrightarrow X$ is *Hölder continuous* with exponent γ, $0 < \gamma < 1$ on $[0, \mathbf{T})$ if there is a constant L such that

$$\|f(t) - f(s)\| \leq L\,|t-s|^{\gamma} \qquad \text{for } 0 \leq s \leq t < \mathbf{T}.$$

It is *locally Hölder continuous* if every $t \in [0, \mathbf{T})$ has a neighbourhood in which f is Hölder continuous.

3. Main results

Definition 3.1. A function $u(t) : [t_0, t_1) \longrightarrow X$ is called a (classical) *solution* of (1) if it satisfies the following three conditions:

(i) $u(t) \in C([t_0, t_1), X) \cap C^1((t_0, t_1), X)$ and $u(t_0) = x_0$.

(ii) $u(t) \in D(A)$ and $(t, u(t)) \in U$ for all $t_0 < t < t_1$.

(iii) $\dfrac{d}{dt}[u(t) + f(t, u(t))] = Au(t) + g(t, u(t))$ for all $t_0 < t < t_1$.

Before going further, let us do some formal analysis to see what a potential solution of (1) should look like. We use the integrating factor method, treating A as a number. The equation

$$\frac{d}{dt}\left[u(t)+f(t,u(t))\right]=Au(t)+g(t,u(t))$$

has integrating factor e^{-tA}. Multiplying by this integrating factor, we have

$$\begin{aligned}
&e^{-tA}\frac{d}{dt}\left[u(t)+f(t,u(t))\right]=e^{-tA}Au(t)+e^{-tA}g(t,u(t))\\
&e^{-tA}\frac{d}{dt}\left[u(t)+f(t,u(t))\right]-e^{-tA}Au(t)=e^{-tA}g(t,u(t))\\
&e^{-tA}\frac{du(t)}{dt}+e^{-tA}\frac{d}{dt}f(t,u(t))-e^{-tA}Au(t)=e^{-tA}g(t,u(t))\\
&e^{-tA}\frac{du(t)}{dt}-e^{-tA}Au(t)=-e^{-tA}\frac{d}{dt}f(t,u(t))+e^{-tA}g(t,u(t))\\
&\frac{d}{dt}\left(e^{-tA}u(t)\right)=-e^{-tA}\frac{d}{dt}f(t,u(t))+e^{-tA}g(t,u(t))\\
&e^{-sA}u(s)|_{t_0}^{t}=\int_{t_0}^{t}-e^{-sA}\frac{d}{ds}f(s,u(s))ds+\int_{t_0}^{t}e^{-sA}g(s,u(s))ds\\
&e^{-tA}u(t)-e^{-t_0A}u(t_0)=\int_{t_0}^{t}-e^{-sA}\frac{d}{ds}f(s,u(s))ds+\int_{t_0}^{t}e^{-sA}g(s,u(s))ds\\
&e^{-tA}u(t)=e^{-t_0A}u(t_0)+\int_{t_0}^{t}-e^{-sA}\frac{d}{ds}f(s,u(s))ds+\int_{t_0}^{t}e^{-sA}g(s,u(s))ds\\
&u(t)=e^{tA}e^{-t_0A}u(t_0)+\int_{t_0}^{t}-e^{tA}e^{-sA}\frac{d}{ds}f(s,u(s))ds+\int_{t_0}^{t}e^{tA}e^{-sA}g(s,u(s))ds\\
&u(t)=e^{(t-t_0)A}u(t_0)+\int_{t_0}^{t}-e^{(t-s)A}\frac{d}{ds}f(s,u(s))ds+\int_{t_0}^{t}e^{(t-s)A}g(s,u(s))ds\\
&u(t)=e^{(t-t_0)A}u(t_0)-f(t,u(t))+e^{(t-t_0)A}f\left(t_0,u(t_0)\right)-\int_{t_0}^{t}Ae^{(t-s)A}f(s,u(s))ds\\
&\qquad+\int_{t_0}^{t}e^{(t-s)A}g(s,u(s))ds
\end{aligned}$$

$$u(t) = T\,(t - t_0)\,x_0 - f(t, u(t)) + T\,(t - t_0)\,f\,(t_0, u(t_0)) - \int_{t_0}^{t} AT\,(t - s)\,f(s, u(s))ds$$
$$+ \int_{t_0}^{t} T\,(t - s)\,g(s, u(s))ds$$
$$u(t) = T\,(t - t_0)\,[x_0 + f\,(t_0, u(t_0))] - f(t, u(t)) - \int_{t_0}^{t} AT\,(t - s)\,f(s, u(s))ds$$
$$+ \int_{t_0}^{t} T\,(t - s)\,g(s, u(s))ds$$

The following lemma will be needed in the proof of our main result.

Lemma 3.2. If $u(t)$ is a solution of (1) on (t_0, t_1), then

$$\begin{aligned} u(t) = T\,(t - t_0)\,[x_0 + f\,(t_0, u(t_0))] - f(t, u(t)) - \int_{t_0}^{t} AT\,(t - s)\,f(s, u(s))ds \\ + \int_{t_0}^{t} T\,(t - s)\,g(s, u(s))ds \end{aligned} \tag{4}$$

Conversely, if $u(t) : [t_0, t_1) \longrightarrow X_\alpha$ is a continuous function, and

$$\int_{t_0}^{t_0+\tau} \|f(s, u(s))\|\,ds < \infty, \qquad \int_{t_0}^{t_0+\tau} \|g(s, u(s))\|\,ds < \infty$$

for some $\tau > 0$, and if the integral equation (4) holds for $t_0 < t < t_1$, then $u(t)$ is a solution of (1) on (t_0, t_1).

Proof. The first claim is immediate in view of the solution above and from Definition 3.1. Suppose $u(t)$ is a solution of the integral equation (4) and $u(t) \in C((t_0, t_1), X_\alpha)$. We first prove that $u(t) : (t_0, t_1) \longrightarrow X_\alpha$ is locally Hölder continuous. If $t_0 < t < t+h < t_1$, then

$$\begin{aligned} u(t+h) - u(t) = {} & T\,(t + h - t_0)\,[x_0 + f\,(t_0, u(t_0))] - f(t + h, u(t + h)) \\ & - \int_{t_0}^{t+h} AT\,(t + h - s)\,f(s, u(s))ds + \int_{t_0}^{t+h} T\,(t + h - s)\,g(s, u(s))ds \\ & - T\,(t - t_0)\,[x_0 + f\,(t_0, u(t_0))] + f(t, u(t)) + \int_{t_0}^{t} AT\,(t - s)\,f(s, u(s))ds \\ & - \int_{t_0}^{t} T\,(t - s)\,g(s, u(s))ds \end{aligned}$$

$$
\begin{aligned}
&= T\,(t+h-t_0)\,[x_0 + f\,(t_0, u(t_0))] - T\,(t-t_0)\,[x_0 + f\,(t_0, u(t_0))] \\
&\quad - f(t+h, u(t+h)) + f(t, u(t)) + \int_{t_0}^{t} AT\,(t-s)\, f(s, u(s))ds \\
&\quad - \int_{t_0}^{t+h} AT\,(t+h-s)\, f(s, u(s))ds + \int_{t_0}^{t+h} T\,(t+h-s)\, g(s, u(s))ds \\
&= T\,(t+h-t_0)\,[x_0 + f\,(t_0, u(t_0))] - T\,(t-t_0)\,[x_0 + f\,(t_0, u(t_0))] \\
&\quad - [f(t+h, u(t+h)) - f(t, u(t))] + \int_{t_0}^{t} AT\,(t-s)\, f(s, u(s))ds \\
&\quad - \int_{t_0}^{t} AT\,(t+h-s)\, f(s, u(s))ds - \int_{t}^{t+h} AT\,(t+h-s)\, f(s, u(s))ds \\
&\quad + \int_{t}^{t+h} T\,(t+h-s)\, g(s, u(s))ds + \int_{t_0}^{t} T\,(t+h-s)\, g(s, u(s))ds \\
&\quad - \int_{t_0}^{t} T\,(t-s)\, g(s, u(s))ds - \int_{t_0}^{t} T\,(t-s)\, g(s, u(s))ds \\
&= T(h)T\,(t-t_0)\,[x_0 + f\,(t_0, u(t_0))] - T\,(t-t_0)\,[x_0 + f\,(t_0, u(t_0))] \\
&\quad - [f(t+h, u(t+h)) - f(t, u(t))] + \int_{t_0}^{t} AT\,(t-s)\, f(s, u(s))ds \\
&\quad - \int_{t_0}^{t} AT(h)T\,(t-s)\, f(s, u(s))ds - \int_{t}^{t+h} AT(h)T\,(t-s)\, f(s, u(s))ds \\
&\quad + \int_{t}^{t+h} T(h)T\,(t-s)\, g(s, u(s))ds + \int_{t_0}^{t} T(h)T\,(t-s)\, g(s, u(s))ds \\
&\quad - \int_{t_0}^{t} T\,(t-s)\, g(s, u(s))ds - \int_{t_0}^{t} T\,(t-s)\, g(s, u(s))ds \\
&= T\,(t-t_0)\,(x_0 + f\,(t_0, u(t_0)))\,[T(h) - I] - [f(t+h, u(t+h)) - f(t, u(t))] \\
&\quad - \int_{t_0}^{t} (T(h) - I)AT\,(t-s)\, f(s, u(s))ds + \int_{t_0}^{t} (T(h) - I)T\,(t-s)\, g(s, u(s))ds \\
&\quad - \int_{t}^{t+h} AT\,(t+h-s)\, f(s, u(s))ds + \int_{t}^{t+h} T\,(t+h-s)\, g(s, u(s))
\end{aligned}
$$

Now if $0 < \delta < 1 - \alpha$, then for any $z \in X$, we have by Lemma 2.2

$$
\|(T(h) - I)T(t-s)z\|_{\alpha} \le C(t-s)^{-(\alpha+\delta)} h^{\delta} e^{a(t-s)}\, \|z\|.
$$

Hence for $t \in [t_0', t_1'] \subset (t_0, t_1)$,

$$
\|u(t+h) - u(t)\|_{\alpha} \le K h^{\delta}, \qquad K \text{ a constant.}
$$

It follows that $t \longrightarrow f(t, u(t)), t \longrightarrow g(t, u(t))$ are locally H$\ddot{o}lder$ continuous on (t_0, t_1). So $u(t)$ solves the linear equation

$$\frac{d}{dt}[v(t) + f(t, v(t))] + Av(t) = g(t, u(t)), \qquad t_0 < t < t_1$$
$$v(t_0) = x_0.$$

Hence, $u(t)$ is also a solution of (1) on (t_0, t_1). ■

Definition 3.3. A continuous solution u of the integral equation (4) is called a *mild solution* of the initial value problem (1).

We can now proceed to the proof of the following theorem which is the main result of this paper.

Theorem 3.4. Let A be the infinitesimal generator of an analytic semigroup $T(t)$, $0 \leq \alpha < 1$, and $f, g : U \longrightarrow X$, U an open set of $[0, \infty) \times X_\alpha$, $f(t, x), g(t, x)$ are locally H$\ddot{o}lder$ continuous in t, locally Lipschitz continuous in x; then for any $(t_0, x_0) \in U$ there exists $\tau = \tau(t_0, x_0) > 0$ such that (1) has a unique mild solution u on $(t_0, t_0 + \tau)$ provided $c < 1$ where

$$c = L_f M_\alpha + A^\alpha L_f + \frac{L_f \delta}{4(B_f + L_f \delta)} + \frac{L_g \delta}{4(B_g + L_g \delta)}$$

and $B = \max_{[t_0, t_0+\epsilon]} \|f(t_0, x_0)\|$.

Proof. It suffices to prove the result for the integral equation (4), the "variation-of-constants formular".

Choose $\delta > 0$, $\epsilon > 0$, such that the set

$$V = \{(t, x) \in [0, \infty) \times X_\alpha | t \leq t \leq t_0 + \epsilon, \|x - x_0\|_\alpha \leq \delta\} \subset U,$$

and

$$\|f(t, x) - f(t, y)\| \leq L \|x - y\|_\alpha \qquad \text{for } (t, x), (t, y) \in V.$$

Now let

$$B = \max_{[t_0, t_0+\epsilon]} \|f(t_0, x_0)\| \tag{5}$$

and choose τ so that $0 < \tau \leq \epsilon$ and

$$\|(T(h) - I)x_0\|_\alpha \leq \delta/4 \qquad \text{for } 0 \leq h \leq \tau, \tag{6}$$

$$M_\alpha(B + L\delta) \int_0^\tau u^{-\alpha} e^{au} du \leq \delta/4 \tag{7}$$

where

$$\left\| A^{\alpha} T(t) \right\| \leq M_{\alpha} t^{-\alpha} e^{at} \qquad \text{for } t > 0. \tag{8}$$

Let S denote the set of continuous functions $u : [t_0, t_0+\tau] \longrightarrow X_{\alpha}$ such that $\|u(t) - x_0\|_{\alpha} \leq \delta$ on $t_0 \leq t \leq t_0 + \tau$, provided with the maximum norm

$$\|u\| = \max_{[t_0, t_0+\tau]} |u(t)|,$$

then S is a complete metric space.

For $u \in S$, we define

$$\Phi u(t) : [t_0, t_0 + \tau] \longrightarrow X \qquad \text{by}$$

$$\Phi u(t) = T\,(t - t_0)\,[x_0 + f\,(t_0, u(t_0))] - f(t, u(t)) - \int_{t_0}^{t} AT\,(t - s)\, f(s, u(s))ds + \int_{t_0}^{t} T\,(t - s)\, g(s, u(s))ds.$$

We show that Φ maps S into itself, i.e.,

$$\Phi : S \longrightarrow S \tag{9}$$

and Φ is a strict contraction.

To show this, it suffices to verify that

$$\|\Phi u(t) - x_0\|_{\alpha} = \left\| \Phi u(t) - A^{\alpha} x_0 \right\| \leq \delta, \qquad t \leq t \leq t_0 + \tau.$$

So

$$\begin{aligned} \|\Phi u(t) - x_0\|_{\alpha} \leq{}& \|(T(t - t_0) - I)x_0\|_{\alpha} + \|T(t - t_0) f(t_0, u(t_0)) - f(t, u)\|_{\alpha} \\ &+ \left\| \int_{t_0}^{t} AT(t - s)\,[f(s, u(s)) - f(s, x_0)]\,ds \right\|_{\alpha} \\ &+ \left\| \int_{t_0}^{t} AT(t - s) f(s, x_0) ds \right\|_{\alpha} \\ &+ \left\| \int_{t_0}^{t} T(t - s)\,[g(s, u(s)) - g(s, x_0) ds] \right\|_{\alpha} + \left\| \int_{t_0}^{t} T(t - s) g(s, x_0) ds \right\|_{\alpha} \end{aligned}$$

$$
\begin{aligned}
&\leq \|(T(t-t_0)-I)x_0\|_\alpha + \|T(t-t_0)f(t_0,u(t_0)) - f(t,u)\|_\alpha \\
&\quad + L_f \left\| \int_{t_0}^{t} A^{\alpha+1}T(t-s)ds \right\| \|u-x_0\| + B_f \left\| \int_{t_0}^{t} A^{\alpha+1}T(t-s)f(s,x_0)ds \right\| \\
&\quad + L_g \left\| \int_{t_0}^{t} A^{\alpha}T(t-s)ds \right\| \|u-x_0\| + B_g \left\| \int_{t_0}^{t} A^{\alpha}T(t-s)ds \right\| \\
&\leq \|(T(t-t_0)-I)x_0\|_\alpha + \|T(t-t_0)f(t_0,u(t_0)) - f(t,u)\|_\alpha \\
&\quad + L_f\delta \left\| \int_{t_0}^{t} A^{\alpha+1}T(t-s)ds \right\| + B_f \left\| \int_{t_0}^{t} A^{\alpha+1}T(t-s)ds \right\| \\
&\quad + L_g\delta \left\| \int_{t_0}^{t} A^{\alpha}T(t-s)ds \right\| + B_g \left\| \int_{t_0}^{t} A^{\alpha}T(t-s)ds \right\| \\
&\leq \frac{\delta}{4} + \left(L_f\delta + B_f\right) \left\| \int_{t_0}^{t} A^{\alpha+1}T(t-s)ds \right\| + \left(L_g\delta + B_g\right) \left\| \int_{t_0}^{t} A^{\alpha}T(t-s)ds \right\| \\
&\leq \frac{\delta}{2} + \left(L_f\delta + B_f\right) M_{\alpha+1} \int_{t_0}^{t_0+\tau} (t-s)^{-\alpha-1} e^{a(t-s)} ds \\
&\quad + \left(L_g\delta + B_g\right) M_\alpha \int_{t_0}^{t_0+\tau} (t-s)^{-\alpha} e^{a(t-s)} ds \qquad \text{by (3.5).}
\end{aligned}
$$

But by (3.4),

$$
\int_{t_0}^{t_0+\tau} (t-s)^{-\alpha} e^{a(t-s)} ds = \int_0^{\tau} u^{-\alpha} e^{au} du \leq \frac{\delta}{4M_\alpha(B+L\delta)}.
$$

So

$$
\begin{aligned}
\|\Phi u(t) - x_0\|_\alpha &\leq \frac{\delta}{2} + \frac{\left(L_f\delta + B_f\right) M_{\alpha+1}\delta}{4M_{\alpha+1}\left(B_f + L_f\delta\right)} + \frac{\left(L_g\delta + B_g\right) M_\alpha\delta}{4M_\alpha\left(B_g + L_g\delta\right)} \\
&\leq \frac{\delta}{2} + \frac{\delta}{2} = \delta, \text{ for } t_0 \leq t \leq t_0 + \tau.
\end{aligned}
$$

Also, $\Phi(y) : [t_0, t_0+\tau] \longrightarrow X_\alpha$ is continuous by the continuity of f, g and strongly continuity of $T(t)$, and so Φ maps S into itself.

We next show that

$$
\|\Phi u(t) - \Phi v(t)\| \leq c\,\|u - v\| \qquad \text{for } u, v \in S,\ c < 1. \tag{10}
$$

For $t_0 \le t \le t_0 + \tau$,

$$\begin{aligned}
\|\Phi u(t) - \Phi v(t)\|_\alpha &\le \|T(t-t_0)\,[f(t_0,u(t_0)) - f(t_0,v(t_0))]\|_\alpha + \|f(t,u(t)) - f(t,v(t))\|_\alpha \\
&\quad + \left\| \int_{t_0}^{t} T(t-s)\,[f(s,u(s)) - f(s,v(s))]\,ds \right\|_\alpha \\
&\quad + \left\| \int_{t_0}^{t} T(t-s)\,[g(s,u(s)) - g(s,v(s))]\,ds \right\|_\alpha \\
&= \left\| A^\alpha T(t-t_0)\,[f(t_0,u(t_0)) - f(t_0,v(t_0))] \right\| \\
&\quad + \left\| A^\alpha\,(f(t,u(t)) - f(t,v(t))) \right\| \\
&\quad + \left\| \int_{t_0}^{t} A^{\alpha+1} T(t-s)\,[f(s,u(s)) - f(s,v(s))]\,ds \right\| \\
&\quad + \left\| \int_{t_0}^{t} A^{\alpha} T(t-s)\,[g(s,u(s)) - g(s,v(s))]\,ds \right\| \\
&\le L_f M_\alpha \|u-v\| + A^\alpha L_f \|u-v\| \\
&\quad + L_f M_{\alpha+1} \int_{t_0}^{t_0+\tau} (t-s)^{-\alpha-1} e^{a(t-s)}\,\|u-v\| \\
&\quad + L_g M_\alpha \int_{t_0}^{t_0+\tau} (t-s)^{-\alpha} e^{a(t-s)}\,\|u-v\| \\
&\le \left[L_f M_\alpha + A^\alpha L_f + L_f M_{\alpha+1} \frac{\delta}{4M_{\alpha+1}(B+L_f\delta)} \right] \|u-v\| \\
&\quad + \left[L_g M_\alpha \frac{\delta}{4M_\alpha(B+L_g\delta)} \right] \|u-v\| \\
&\le \left[L_f M_\alpha + A^\alpha L_f + \frac{L_f\delta}{4(B+L_f\delta)} + \frac{L_g\delta}{4(B+L_g\delta)} \right] \|u-v\|
\end{aligned}$$

for all $u, v \in S$.

Clearly, if $c = L_f M_\alpha + A^\alpha L_f + \dfrac{L_f\delta}{4(B+L_f\delta)} + \dfrac{L_g\delta}{4(B+L_g\delta)} < 1$, then by (3.6) and (3.7), one can apply the contraction mapping theorem to Φ to obatain that

There exists a unique fixed point $u \in S$, which is a continuous solution of the integral equation (4) and has $f(t,u(t)), g(t,u(t))$ bounded as $t \to t^+$. By Lemma 3.2, this is the unique solution of (1) on $(t_0, t_0+\tau)$. ■

Example 3.5. Consider

$$\frac{\partial}{\partial t}\,[u(t,x) + f(t,x,u(t,x))] = u\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x} + g(t,x,u(x,t)) \qquad 0 < x < 1, t > 0 \tag{11}$$

$$u(0,t) = 0, u(1,t) = 0 \qquad 0 < x < 1, t > 0 \tag{12}$$

where $f, g : \mathbb{R}^+ \times [0, \pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ are measurable in x, locally Hölder continuous in t and locally Lipschitz continuous in u, uniformly in x, with

$$|g(t, x, u)| \leq h(x) r(t, |u|), \qquad h \in L^2(0, 1), \tag{13}$$

r continuous and increasing in its second argument.

Take

$$X = L^2(0, 1), A = -\frac{d^2}{dx^2}$$

with domain

$$D(A) = H^2(0, 1) \cap H_0^1(0, 1)$$

and

$$D(A^{1/2}) = X_{1/2} = H_0^1(0, 1).$$

We write (3.8) as

$$F(t, \phi)(x) = -\phi(x)(\phi)'(x) + g(t, x, \phi(x)) \tag{14}$$

and prove that

$$F : \mathbb{R}^+ \times H_0^1(0, 1) \longrightarrow L^2(0, 1), \qquad 0 < x < 1,$$

satisfies the hypotheses of Theorem 3.4.

Now, we observe that, for $\phi \in H_0^1(0, 1)$, $\phi(x) = \int_0^x \phi'(\xi) d\xi$, so ϕ is absolutely continuous with

$$\sup_x |\phi(x)| \leq \|\phi\|_{1/2},$$

so from (3.10) and (3.11), we have

$$\|F(t, \phi)\|_{L^2(0,1)} \leq \|\phi\|_{1/2}^2 + \|h\|_{L^2(0,1)} r\left(t, \|\phi\|_{1/2}\right),$$

and F maps bounded sets in $\mathbb{R}^+ \times X_{1/2}$ into bounded sets in X. Also, if $(t_0, \phi_0) \in \mathbb{R}^+ \times X_{1/2}$, ϕ_0 continuous, then there exists a neighbourhood V of the compact set $\{(t_0, x, \phi_0(x)) : 0 \leq x \leq 1\}$ in $\mathbb{R}^+ \times [0, 1] \times \mathbb{R}$ and positive constants L, θ so that for $(t_1, x, u_1) \in V$, $(t_2, x, u_2) \in V$,

$$|g(t_1, x, u_1) - g(t_2, x, u_2)| \leq L\left(|t_1 - t_2|^\theta + |u_1 - u_2|\right).$$

Hence there is a neighbourhood U of (t_0, ϕ_0) in $\mathbb{R}^+ \times X_{1/2}$ so that

$$(t, \phi) \in U \Rightarrow (t, x, \phi(x)) \in V \qquad \text{for a.e } 0 \leq x \leq 1;$$

and if $(t_1, \phi_1) \in U$, $(t_2, \phi_2) \in U$ we have

$$|g(t_1, \cdot, \phi_1(\cdot)) - g(t_2, \cdot, \phi_2(\cdot))|_{L^2(0,1)} \leq L\left(|t_1 - t_2|^\theta + |\phi_1 - \phi_2|\right).$$

Also for any ϕ_1, ϕ_2 in $X_{1/2}$,

$$\begin{aligned}\left\|\phi_1\phi_1' - \phi_2\phi_2'\right\|_{L^2(0,1)} &\leq \left\|\phi_1\left(\phi_1' - \phi_2'\right)\right\|_{L^2} + \left\|(\phi_1 - \phi_2)\,\phi_2'\right\|_{L^2} \\ &\leq \|\phi_1\|_{1/2}\,\|\phi_1 - \phi_2\|_{1/2} + \|\phi_2\|_{1/2}\cdot\|\phi_1 - \phi_2\|_{1/2} \\ &\leq \left(\|\phi_1\|_{1/2} + \|\phi_2\|_{1/2}\right)\|\phi_1 - \phi_2\|_{1/2}.\end{aligned}$$

Thus all the hypotheses of Theorem 3.4 are verified for the boundary value problem (3.10)-(3.11).

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