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A Generalized Coding Theorem in Terms of Useful Information Measures

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Abstract

In this paper a generalized 'useful' parametric mean length $L_R\left(P^{\nu},U\right)$ has been defined and bounds for $L_R\left(P^{\nu},U\right)$ are obtained in terms of generalized useful R-norm information measure.

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1. Introduction

Consider the model A given below for a finite scheme random experiment having $\left(A_1, A_2,..., A_n\right)$ as the complete system of events with respective probabilities $P=\left(p_1, p_2,.., p_n\right), p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$ and credited with utilities $U=\left(u_1,u_2,...,u_n\right)$, $u_i > 0$, $i = 1,2,...,n$ Denote

$$A=\begin{bmatrix} A_1 & A_2 & & A_n \\ p_1 & p_2 & & p_n \\ u_1 & u_2 & & u_n \end{bmatrix} \tag{1.1}$$

We call the scheme (1.1) as a finite information scheme. Every finite scheme describes a state of uncertainty. Shannon [6] introduced a quantity which in a reasonable way, measures the amount of uncertianity (entropy). This measure is given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{1.2}$$

Can serve as a very suitable measure of entropy of the finite scheme . Through out the paper, logarithms are taken to base $D(D \geq 2)$.

Also, Guiasu and Picard [3] introduced a quantity in terms of utilities which also measure the amount of uncertianity associated with a given finite scheme. This measure is given by

$$H(P,U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} \tag{1.3}$$

Let $X = (x_1, x_2, ..., x_n)$ be the finite set of input symbols which are to be encoded using alphabet of D symbols. It has been shown Feinstein [2] that there is a unique decipherable code with lengths $l_1, l_2, ..., l_n$ and satisfying

$$\sum_{i=1}^{n} D^{-l_i} \leq 1 \tag{1.4}$$

where D is the size of the code alphabet.

Noiseless coding theorem for Shannon's entropy with ordinary code mean length

$$L = \sum_{i=1}^{n} l_i p_i \tag{1.5}$$

under the condition (1.4),has played an important role in ordinary communication theory, (Shannon [6]).

Khan and Haseen [4], Khan, Autar and Haseen [5], Boekee et al [1] and Singh, Kumar and Tuteja [8] have studied generalized coding theorems by considering different generalized measures of (1.2) and (1.5) under the condition (1.4) of unique decipherability.

In this paper, we study coding theorems by considering a new function depending on the parameters R and ν. Our motivation for studying this new function is that it generalizes some entropy functions already existing in the literature.

2. Coding theorems

Consider a function

$$H_R\left(P^{\nu},U\right)=\frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^{n}u_i p_i^{R+\nu-1}}{\sum_{i=1}^{n}u_i p_i^{\nu}}\right)^{\frac{1}{R}}\right] \tag{2.1}$$

for all $R\in\Re_+(\neq 1),\nu\neq 1\ ,\sum_{i=1}^{n}p_i=1\ ,\ i=1,2,...,n$

(i) When $\nu=1$, (2.1) reduces to the useful R-norm information due Singh, Kumar and Tuteja [8].

(ii)When $\nu=1,u_i=1\ \forall\ i=1,2,...,n$, (2.1) reduces to the R-norm information measure due to Boekee et al [1].

(iii) When $R\to 1,\nu=1$ and $u_i=1\ \forall\ i=1,2,...n$. (2.1) reduces to the measure due to Shannon [6].

Further consider

$$L_R\left(P^{\nu},U\right)=\frac{R}{R-1}\left[1-\frac{\sum_{i=1}^{n}u_i p_i^{\nu}D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n}u_i p_i^{\nu}}\right] \tag{2.2}$$

where $R\in\Re_+\ ,\ R\neq 1$.

(i)For $\nu=1$, (2.2) reduces to the mean length due to Singh, Kumar and Tuteja [8].

(ii)For $\nu=1,\ u_i=1\ \forall\ i=1,2,...n$. (2.2) reduces to the mean length due to Boekee et al [1].

(iii)For $R\to 1,\nu=1,\ u_i=1,$, (2.2) reduces to the optimal code length defined by Shannon [6].

We now establish a result, that in a sense, gives a characterization of $H_R\left(P^{\nu},U\right)$ under the condition

(2.3) $\sum_{i=1}^{n}u_i p_i^{\nu-1}D^{-l_i}\leq\sum_{i=1}^{n}u_i p_i^{\nu}$

Remark: For $\nu=1,u_i=1\ \forall i=1,2,...n$ and $\sum_{i=1}^{n}p_i=1$, (2.3) is a generalization of (1.4).

Theorem 1: For every code whose lengths $l_1, l_2, ..., l_n$ satisfies (2.3), the average length satisfies

(2.4) $L_R(P^{\nu}, U) \geq H_R(P^{\nu}, U)$

equality holds if and only if

$$l_i = -\log \frac{u_i p_i^{R}}{\left(\dfrac{\sum_{i=1}^{n} u_i p_i^{R+\nu-1}}{\sum_{i=1}^{n} u_i p_i^{\nu}} \right)} \tag{2.5}$$

Proof: we use Holders inequality [7]

$$\sum_{i=1}^{n} x_i y_i \geq \left[\sum_{i=1}^{n} x_i^{p} \right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} y_i^{q} \right]^{\frac{1}{q}} \tag{2.6}$$

for all $x_i > 0$, $y_i > 0$, $i = 1,2,...,n$, $p < 1 (\neq 0)$ and $p^{-1} + q^{-1} = 1$
with equality if and only if there exists a positive number c such that

$$x_i^{p} = c\, y_i^{q} \tag{2.7}$$

Setting

$x_i = u_i p_i^{-\frac{\nu R}{1-R}} D^{-l_i}$

$y_i = u_i p_i^{\frac{R+\nu-1}{1-R}}$

$P = \dfrac{R-1}{R}$ and $q = 1 - R$ in (2.6) and using (2.3), Also if $R > 1$ we get

$$\left[\sum_{i=1}^{n} u_i p_i^{\nu} D^{-l_i\left(\frac{R-1}{R}\right)} \right]^{\frac{R}{1-R}} \geq \frac{\left[\sum_{i=1}^{n} u_i p_i^{R+\nu-1} \right]^{\frac{1}{1-R}}}{\sum_{i=1}^{n} u_i p_i^{\nu}} \tag{2.8}$$

Dividing both sides of (2.8) by $\left(\sum_{i=1}^{n} u_i p_i^{\nu} \right)^{\frac{R}{1-R}}$, we get

$$\left[\frac{\sum_{i=1}^{n} u_i p_i^{\,v} D^{-l_i\left(\frac{R-1}{R}\right)}}{\sum_{i=1}^{n} u_i p_i^{\,v}}\right]^{\frac{R}{1-R}} \geq \left[\frac{\sum_{i=1}^{n} u_i p_i^{\,R+v-1}}{\sum_{i=1}^{n} u_i p_i^{\,v}}\right]^{\frac{1}{1-R}}$$

Raising both sides to the power $\frac{1-R}{R}$, $R \neq 1$ also $\frac{R}{R-1} > 0$ for $R > 1$and after suitable operations, we obtain the result (2.4). For $0 < R < 1$, the inequality (2.4) can be proved in a similar fashion.

Theorem 2: For every code with lengths $l_1, l_2, ..., l_n$ satisfies (2.3). $L_R(P^v, U)$ can be made to satisfy the inequality

$$L_R(P^v, U) < H_R(P^v, U) D^{\frac{1-R}{R}} + \frac{R}{R-1}\left(1 - D^{\frac{1-R}{R}}\right) \tag{2.9}$$

Proof: Let l_i be the positive integer satisfying the inequality

$$-\log \frac{u_i p_i^{\,R}}{\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R+v-1}}{\sum_{i=1}^{n} u_i p_i^{\,v}}\right)} \leq l_i < -\log \frac{u_i p_i^{\,R}}{\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R+v-1}}{\sum_{i=1}^{n} u_i p_i^{\,v}}\right)} + 1 \tag{2.10}$$

Consider the interval

$$\delta_i = \left[-\log \frac{u_i p_i^{\,R}}{\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R+v-1}}{\sum_{i=1}^{n} u_i p_i^{\,v}}\right)}, -\log \frac{u_i p_i^{\,R}}{\left(\frac{\sum_{i=1}^{n} u_i p_i^{\,R+v-1}}{\sum_{i=1}^{n} u_i p_i^{\,v}}\right)} + 1\right] \tag{2.11}$$

of length 1. In every δ_i, there lies exactly one positive number l_i such that

$$0 < -\log \frac{u_i p_i^{\ R}}{\left(\dfrac{\sum_{i=1}^{n} u_i p_i^{\ R+\nu-1}}{\sum_{i=1}^{n} u_i p_i^{\ \nu}} \right)} \le l_i < -\log \frac{u_i p_i^{\ R}}{\left(\dfrac{\sum_{i=1}^{n} u_i p_i^{\ R+\nu-1}}{\sum_{i=1}^{n} u_i p_i^{\ \nu}} \right)} + 1 \tag{2.12}$$

We will first show that sequence $\{l_1, l_2, ..., l_n\}$, thus defined satisfies (2.3), from (2.12) we have

$$-\log \frac{u_i p_i^{\ R}}{\left(\dfrac{\sum_{i=1}^{n} u_i p_i^{\ R+\nu-1}}{\sum_{i=1}^{n} u_i p_i^{\ \nu}} \right)} \le l_i$$

$$-\log \frac{u_i p_i^{\ R}}{\left(\dfrac{\sum_{i=1}^{n} u_i p_i^{\ R+\nu-1}}{\sum_{i=1}^{n} u_i p_i^{\ \nu}} \right)} \le -\log_D D^{-l_i}$$

$$\frac{u_i p_i^{\ R}}{\left(\dfrac{\sum_{i=1}^{n} u_i p_i^{\ R+\nu-1}}{\sum_{i=1}^{n} u_i p_i^{\ \nu}} \right)} \ge D^{-l_i} \tag{2.13}$$

Multiplying both sides by $\sum_{i=1}^{n} u_i p_i^{\ \nu-1}$ and summing over $i = 1,2,...,n$, we get (2.3).

The last inequality in (2.12) gives

$$l_i < -\log \frac{u_i p_i^{\ R}}{\left(\dfrac{\sum_{i=1}^{n} u_i p_i^{\ R+\nu-1}}{\sum_{i=1}^{n} u_i p_i^{\ \nu}} \right)} + 1$$

$$l_i < -\log \frac{u_i {p_i}^R}{\left(\dfrac{\sum_{i=1}^{n} u_i {p_i}^{R+\nu-1}}{\sum_{i=1}^{n} u_i {p_i}^{\nu}} \right)} + \log_D D$$

i.e.,

$$D^{-l_i} < \frac{u_i {p_i}^R}{\left(\dfrac{\sum_{i=1}^{n} u_i {p_i}^{R+\nu-1}}{\sum_{i=1}^{n} u_i {p_i}^{\nu}} \right)} D^{-1}$$

or $$D^{-l_i\left(\frac{R-1}{R}\right)} < \left\{ \frac{u_i {p_i}^R}{\left(\dfrac{\sum_{i=1}^{n} u_i {p_i}^{R+\nu-1}}{\sum_{i=1}^{n} u_i {p_i}^{\nu}} \right)} \right\}^{\frac{R-1}{R}} D^{\frac{1-R}{R}}$$

Multiplying both sides by $\dfrac{u_i {p_i}^{\nu}}{\sum_{i=1}^{n} u_i {p_i}^{\nu}}$ and summing over $i = 1,2,...,n$ and simplifying, gives (2.9). For $0 < R < 1$, the proof of the upper bound of $L_R\left(P^{\nu}, U\right)$ follows along the similar lines.

As $D \geq 2$, we have $\dfrac{R}{R-1}\left[1 - D^{\frac{(1-R)}{R}}\right] > 1$ from which it follows that the upper bound of $L_R\left(P^{\nu}, U\right)$ in (2.9) is greater than unity.

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