On the Binding Number of Middle Graph of Graphs

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Abstract

The binding number of a graph was introduced by D.R. Woodall in 1973 [10] and is defined as the minimum of the ratios $|\Gamma(X)|/|X|$ taken over all non-empty subsets of $X$ of vertices in $G$ such that $\Gamma(X) \neq V(G)$, where $\Gamma(X) = \cup_{v \in X} \Gamma(v)$ and $\Gamma(v)$ the set of all vertices adjacent to a vertex $v$ in $G$. We obtain exact values of the binding number of middle graph of cycle, path, complete graph and complete bipartite graph.

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1. Introduction

We consider only finite simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. For a graph $G = (V, E)$ and a set $X \subseteq V$, we denote by $\Gamma(X)$ the set of vertices joined to vertices in $X$. A set of independent edges which cover all vertices of a graph is called 1-factor of a graph. By (1,2)-factor of a graph $G$, we mean a set of independent edges or vertex disjoint cycles which cover all vertices of $G$. Clearly, the cycles in the definition are of odd length. A graph $G$ is hallian, if $|\Gamma(X)| \geq |X|$ for any set $X \subseteq V$ or equivalently if $G$ has a (1,2)-factor [2]. Clearly, $G$ is a hallian graph if its vertices can be covered by a set of vertex disjoint even paths or odd cycles. A graph $G$ is k-hallian, if for any set $A$ of vertices of order at most $k$, the subgraph of $G$ induced by the set $V - A$ is hallian. The largest $k$ such that $G$ is $k$-hallian is called the hallian index of $G$ and is denoted by $h(G)$. Clearly $h(G) \leq \delta(G) - 1$ where $\delta(G)$ denotes the minimum degree among the vertices.
of $G$. The middle graph $[1]$ of a graph $G = (V, E)$ denoted by $M(G)$ is a graph with vertex set $V \cup E$, and two vertices in $M(G)$ are adjacent if one is a vertex and other one is an edge incident with it in $G$ or both are adjacent edges in $G$. The binding number of $G$ is defined by D.R. Woodall [10] as,

$$bind(G) = \min_{X} \frac{\left| \Gamma(X) \right|}{|X|}$$

where $\sum$ is the set of all admissible sets of $G$ and $\Gamma(X) \neq V(G)$. The binding number was intensively studied by [4–6]. If $bind(G)$ is large, then $G$ has edges fairly well distributed. Clearly $bind(G) = 0$ if and only if $G$ has an isolated vertex.

2. Existing Results

We state some existing results without proof that are required for establishing the result in this paper.

Proposition 2.1. [6] If $H$ is a spanning subgraph of $G$ then $bind(H) \leq bind(G)$.

Proposition 2.2. [6] If $G$ has a 1-factor then $bind(G) \geq 1$.

Theorem 2.3. [10] If $P_n$ is a path on $n$ vertices then

$$bind(P_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{n - 1}{n + 1} & \text{if } n \text{ is odd} \end{cases}$$

Theorem 2.4. [10] $bind(K_n) = n - 1, n \geq 1$.

Proposition 2.5. [2] For any graph $G$, $h(G) \leq \delta(G) - 1$.

Theorem 2.6. [2] If $G$ is $l$-connected and $k$-hallian, then $|\Gamma(X)| \geq |X| + r$ where $r = \min \{k, l\}$.

Lemma 2.7. [2] If a graph $G$ on $n$ vertices has $h(G) = \delta(G) - 1$ and $k(G) \geq h(G)$, then $bind(G) = \frac{n - 1}{n - \delta(G)}$.

3. Results

In this section we give the exact values of the binding numbers of middle graphs of some well-known classes of graphs, namely, unicyclic graphs, cycles, paths, complete graphs, complete bipartite graphs.

Proposition 3.1. If $G$ is unicyclic graph then $bind(M(G)) = 1$. 
Proof. Let $G$ be a unicyclic graph. Label the vertices and edges of $G$ as $u_1, u_2, \ldots, u_n$ and $e_1, e_2, \ldots, e_n$ in such a way that $u_i$ is on $e_i$ for $i = 1, 2, \ldots, n$. Because of the existence of a one-to-one correspondence between the vertices and edges of $G$, this labeling is possible. Then by the structure of the middle graph $M(G)$ of $G$, the existence of 1-factor in $M(G)$, namely the edges of the form $u_ie_i$ for $i = 1, 2, \ldots, n$ is evident. Then by proposition 2.2 [6], $bind(M(G)) \geq 1$.

Next choose $X = V(G)$ as a subset of $V(M(G))$ then
\[
\left| \frac{\Gamma_{M(G)}(X)}{|X|} \right| = 1
\]
and thus $bind(M(G)) \leq 1$ and hence the result. ■

Corollary 3.2. $bind(M(C_n)) = 1$.

Proposition 3.3. $bind(M(P_n)) = \frac{n-1}{n}$, for $n \geq 1$.

Proof. Label the vertices and edges of $P_n$ as $u_1, u_2, \ldots, u_n$ and $e_1, e_2, \ldots, e_n$ such that $e_i = u_{i}u_{i+1}$ for $i = 1, 2, \ldots, n - 1$. The middle graph $M(P_n)$ of $P_n$ contains $P_{2n-1}$ a path on $2n-1$ vertices as a spanning subgraph. Hence by Proposition 2.1 [6] and Proposition 2.3 [10] we have
\[
bind(M(P_n)) \geq bind(P_{2n-1}) = \frac{2n - 1 - 1}{2n - 1 + 1} = \frac{n - 1}{n}.
\]
Next by choosing $X = \{u_1, u_2, \ldots, u_n\}$ a subset of $V(M(P_n))$ we have
\[
\Gamma_{M(P_n)}(X) = \{e_1, e_2, \ldots, e_{n-1}\},
\]
Thus
\[
bind(M(P_n)) \leq \frac{|\Gamma_{M(P_n)}(X)|}{|X|} = \frac{n - 1}{n}
\]
Combining these two we get the required Result. ■

Theorem 3.4.

\[
bind(M(K_n)) = \begin{cases} 
0 & \text{if } n = 1 \\
1/2 & \text{if } n = 2 \\
1 & \text{if } n = 3 \\
\frac{n^2 + n - 2}{n^2 - n + 2} & \text{if } n \geq 4
\end{cases}
\]

This Theorem requires Lemma 3.5 involving use of another graph theoretic parameter called hallian index of a graph introduced by M.Borowiecki and D. Michalak [2].

Lemma 3.5. $h(M(K_n)) = n - 2, n \geq 4$. 

Assume $X \subseteq V(M(K_n))$, $|X| \leq 2$ a graph $M(K_4) - X$ is hallian (i.e., it has (1-2)-factor). Moreover, if we take a set $X$ containing three vertices $e_{i_1}, e_{i_2}, e_{i_3}$ which correspond to the edges of $K_4$ incident to a vertex $u_i$; the graph $M(K_4) - X$ is not hallian. Thus $h(M(K_4)) = 2$.

Assume that $h(M(K_n)) = n - 2$ for any $n \geq 4$. Let us label vertices of $M(K_{n+1})$ in the following way: the vertices of $X \cap M$ to a vertex $u$ a cycle, $(F)$ an cycle $M$ of $K_n$ some $M_k$ holds $k(n+1)$ and let $Y = \{u_{n+1}\} \cup E'$. By the definition of the middle graph we have the following simple observations:

(a) Each vertex $u_i$ together with vertices adjacent to it, induce complete graph on $n+1$ vertices.

(b) Each vertex $u_i$ for $1 \leq i \leq n$ is adjacent to exactly one vertex in the set $E'$ we denote it by $e_i$.

(c) Every vertex $e_i$ of $E$ is adjacent to exactly two vertices of $E'$.

Assume $X \subseteq V(M(K_{n+1}))$, $|X| = n - 1$ and consider two cases: $X \cap Y \neq \phi$ or $X \cap Y = \phi$. In the first case let $X \subseteq V(M(K_n)) \cup Y$, then $M(K_n) - X$ is hallian, by the induction hypothesis. By (a), a graph $[Y - X]$ is complete on at least two vertices, so it also has an (1-2)-factor. Thus $M(K_{n+1}) - X$ is hallian. In the second case $M(K_n) - X$ where $|X'| = |X| - 1$, has an (1-2)-factor, by the induction hypothesis.

Let $X' = X - \{x\}$ and $F$ be an (1-2)-factor of $M(K_n) - X'$. If $x$ is contained in an odd cycle of $F$, then it is obvious that $M(K_n) - X$ has an (1-2)-factor and the set $Y$ is covered by a cycle then in this case $M(K_{n+1}) - X$ is hallian. If $x$ is contained in an even cycle $C$ then we can cover the vertices of $C \cup \{x, y\}$ by an 1-factor. In the case when $y = u_j$, then we can cover $y$ by an edge $\{y, e_j\}$ (b) and the vertices of $Y - \{e_j\}$, by a cycle, $(Y - \{e_j\})$ induces a complete graph on $n$ vertices, then we have (1-2)-factor of $M(K_{n+1}) - X$. If $y = e_j$ then we can cover $y$ by an edge $\{y, e_j\}$ (c), vertices of $Y - \{e_j\}$ by a cycle, then in this case also $M(K_{n+1}) - X$ is hallian. If $x$ is contained in some $K_2 = \{x, y\}$ of $F$ then we can have (1-2)-factor of $M(K_{n+1}) - X$ in the same way as in above case. Let $X = E'$, then the vertex $u_{n+1}$ is isolated in $M(K_{n+1}) - X$. Thus $M(K_{n+1}) - X$ is not hallian. Finally $h(M(K_{n+1})) = n - 1$.

4. Proof of Theorem 3.4

If $n = 1, 2, 3$ the result follows from Theorem 2.4 [10], Theorem 2.3 [10], Corollary 3.2, above respectively. Further we have $h(M(K_n)) = n - 2 = \delta(M(K_n)) - 1$ and $k(M(K_n)) = n - 1$. Thus by Lemma 2.7 [2] and Lemma 3.5, the result follows.

We define the binding number of a middle graph of complete bipartite graph, using the same method as in Theorem 3.4.
Theorem 4.1.

\[
bind(M(K_{m,n})) = \begin{cases} 
\frac{n}{n+1} & \text{, if } m = 1, n \geq 2 \\
1 & \text{, if } m=n=1, m=n=2 \\
\frac{mn+m+n-1}{mn+n} & \text{, if } m \geq 2, n \geq 3, m \leq n 
\end{cases}
\]

Lemma 4.2. If \( G = M(K_{m,n}) \) then \( h(G) = m - 1 \) for \( m \geq 2, n \geq 3 \) and \( m \leq n \).

Proof. If \( V_1 \) and \( V_2 \) be the partite sets of \( K_{m,n} \) with \( V_1 = \{u_1, u_2, \ldots, u_m\} \) and \( V_2 = \{v_1, v_2, \ldots, v_n\} \). By the structure of \( G \); \( G \) contains line graph \( L(K_{m,n}) \) as an induced subgraph.

\( L(K_{m,n}) \) can be viewed as a cartesian product of \( K_m \) and \( K_n \) and thus the vertices of \( L(K_{m,n}) \) can be arranged in \( m \)-rows and \( n \) columns. After this arrangement the vertices \( u_1, u_2, \ldots, u_m \) can be placed in the first column and the vertices \( v_1, v_2, \ldots, v_n \) in the last row and now we can add necessary edges so as to form \( M(K_{m,n}) \). Clearly each row induces \( K_{n+1} \) as an induced subgraph except the vertices \( v_1, v_2, \ldots, v_n \) in the last row and each column induces \( K_{m+1} \) as an induced subgraph except the vertices \( u_1, u_2, \ldots, u_m \) in the first column.

Let \( A \) be the set of \( m-1 \) vertices of \( G \) by choosing \( l_i \) vertices from \( i^{th} \) row where \( i = 1, 2, 3, \ldots, m \) and \( l_j \) vertices from the last row such that \( l_1 + l_2 + \cdots + l_k = m - 1 \) and \( l_i, l_j \geq 0 \). The removal of \( l_i \) vertices from any row (or column) results in to an induced complete subgraph in the same row (or column) and even paths.

Thus \( G - A \) is a hallian and \( h(G) \geq m - 1 \) but \( \delta(G) = m \) so that \( h(G) \leq \delta(G) - 1 = m - 1 \) which gives us \( h(G) = m - 1 \).

\[ \square \]

Proposition 4.3. In \( M(K_{m,n}) \), \(|\Gamma(X)| \geq |X| + m - 1 \) for every \( X \) such that \(|\Gamma(X)| \neq V(M(K_{m,n})) \) and \( m \leq n, m \geq 2, n \geq 3 \).

Proof. By the structure of \( M(K_{m,n}) \) it is not difficult to see that removal of any \( m-1 \) vertices results into a connected graph and hence \( M(K_{m,n}) \) is \((m-1)\)-conned. By the above Lemma \( M(K_{m,n}) \) is \((m-1)\)-hallian and by the proposition 2.6 [2] \(|\Gamma(X)| \geq |X| + m - 1 \) holds.

\[ \square \]

5. Proof of the Theorem 4.1

To prove the Theorem we consider three cases.

Case 1:
Let \( m = 1, n \geq 2 \). By labelling the vertices and edges of \( K_{1,n} \) as \( u_1 \), to be the center; \( v_1, v_2, \ldots, v_n \) as end vertices and \( e_1, e_2, \ldots, e_n \) as edges we get \( M(K_{1,n}) \) as shown in the following figure.

Let \( Y \subseteq A \cup B \cup C \) be the admissible set such that \( Y \) contains atleast one element of \( A, B, C \) and not more than \( n-1 \) elements of the form \( e_i \) in \( M(K_{1,n}) \). Otherwise \( \Gamma(X) = V(M(K_{1,n})) \). Consider \( X_1 = \{v_1, v_2, \ldots, v_n\}, X_2 = \{u_1\} \) and \( X_3 = \{e_1, e_2, \ldots, e_{n-1}\} \).
1. For $Y \subseteq X_1$, $Y \subseteq X_1 \cup X_2$, $Y \subseteq X_3$ and $Y \subseteq X_1 \cup X_3$ we get respectively $|\Gamma(Y)| = |Y|, |\Gamma(Y)| = n, |\Gamma(Y)| = n + 1 + |Y|$ and $|\Gamma(Y)| = n$.

2. $Y \subseteq X_1 \cup X_3$ that is $Y = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}, e_{j_1}, e_{j_2}, \ldots, e_{j_l}\}$.
   
   (a) If $e_{j_1}, e_{j_2}, \ldots, e_{j_l}$ are not incident to any of the $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ Then $|\Gamma(Y)| = n + 1 + n - k = 2n + l - k, 1 \geq k \leq n - 1$.

   (b) If some $v_{i_r}$'s are incident with $e_{j_r}$'s. Without loss of generality that $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ are incident with $e_{i_1}, e_{i_2}, \ldots, e_{i_l} \in e_{j_1}, e_{j_2}, \ldots, e_{j_l}$ Then $|\Gamma(Y)| = n + l + t$.

3. If $Y \subseteq X_2 \cup X_3$ that is $Y = \{u_1, e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$, Then $|\Gamma(Y)| = n + l + k$.

4. If $Y \subseteq X_1 \cup X_2 \cup X_3$ that is $Y = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}, u_1, e_{j_1}, e_{j_2}, \ldots, e_{j_l}\}$.
   
   (a) If $e_{j_1}, e_{j_2}, \ldots, e_{j_l}$ are incident with any of $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ Then $|\Gamma(Y)| = n + l + n - k = 2n + l - k$.

   (b) If some $v_{i_r}$'s are incident with $e_{j_r}$'s. Then $|\Gamma(Y)| = n + l + t$. Thus,

   $$
   \text{bind}(M(K_{1,n})) = \min \left\{ 1, \frac{n}{n+1}, \frac{n+1+|Y|}{n-1}, \frac{2n+l-k}{k+1}, \frac{n+l+t}{k+1}, \frac{n+1+k}{k+1} \right\} = n/n + 1
   $$

Case 2:
Let $m = 2$ and $n = 1$ then $K_{2,2} = C_4$ and hence by the Corollary 3.2, $\text{bind}(M(K_{2,2})) = 1$, $M(K_{1,1}) = P_3$ so by Theorem 2.3 [10], $\text{bind}(M(K_{1,1})) = 1/2$

Case 3:
If $m \geq 2$, $n \geq 3$, $m \leq n$, then the result can be proved using Lemma 4.2, Proposition 4.3 ($k(G) \geq h(G)$), Lemma 2.7 [2], the result follows.

References


