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Ulam - Hyers, Ulam - Trassias, Ulam - Jrassias Stabilities of an Additive Functional Equation in Generalized 2-Normed Spaces: Direct and Fixed Point Approach

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Abstract

In this paper, the authors established the Ulam - Hyers, Ulam - TRassias and Ulam - JRassias stabilities of the additive functional equation

$$g(x)+g(y+z)=g(x+y)+g(z)$$

in Generalized 2- normed spaces using direct and Fixed point method.

1 INTRODUCTION

The stability problem of functional equations originated from a question of S.M. Ulam [21] concerning the stability of group homomorphisms. D.H. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [20] for linear mappings by considering an unbounded Cauchy difference.

The paper of Th.M. Rassias [20] has provided a lot of influence in the development of what we call **Ulam - TRassias stability** of functional equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1982, J.M. Rassias [16] followed the innovative approach of the Th.M. Rassias theorem [20] in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\|y\|^q$ for $p,q\in R$ with $p+q=1$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [19] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. This stability is now called **Ulam -JRassias stability** of functional equations. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 7, 10, 11, 12]).

The solution and stability of the following additive functional equations

$$g(x+y) = g(x)+g(y), \tag{1.1}$$

$$g(2x-y)+g(x-2y) = 3g(x)-3g(y), \tag{1.2}$$

$$g(x+y-2z)+g(2x+2y-z) = 3g(x)+3g(y)-3g(z), \tag{1.3}$$

$$g(2x\pm y\pm z) = g(x\pm y)+g(x\pm z), \tag{1.4}$$

were discussed in [1, 13, 18, 3]. Also M. Arunkumar et. al., [5] investigated the generalized Ulam-Hyers stability of a functional equation

$$f(y) = \frac{f(y+z)+f(y-z)}{2}$$

which is originating from arithmetic mean of consecutive terms of an arithmetic progression using direct and fixed point methods.

Recently, M.Arunkumar, P.Agilan [6] established the solution and stability of the following additive functional equation and inequality

$$f(x)+f(y+z)-f(x+y) = f(z) \tag{1.5}$$

and

$$\| f(x)+f(y+z)-f(x+y) \| \leq \| f(z) \|. \tag{1.6}$$

in Banach space in the sense of Ulam, Hyers, Rassias.

In this paper, the authors established the solution and generalized Ulam-Hyers stability of the additive functional equation

$$g(x)+g(y+z) = g(x+y)+g(z) \tag{1.7}$$

in Generalized 2- normed spaces.

In Section 2, we present some basic definitions and notations in generalized 2-normed spaces. In Section 3, the generalized Ulam-Hyers stability of the functional equation (1.7) is investigated using direct method. The generalized Ulam-Hyers stability of the functional equation (1.7) using fixed point approach is established in Section 4.

2 PRELIMINARIES

In this section, the authors present some basic definitions and notations related to Generalized 2-normed spaces.

Definition 2.1 *[4] Let X be linear space. A function $N(.,.): X \times X \to [0,\infty)$ is called a generalized 2-normed space if it satisfies the following*

(G1) $N(x, y) = 0$ if and only if x and y are linearly independent vectors.

(G2) $N(x, y) = N(y, x)$ for all $x, y \in X$,

(G3) $N(\lambda x, y) =|\lambda| N(x, y)$ for all $x, y \in X$ and $X = \varphi, \varphi$ is a real or complex field,

(G4) $N(x + y, z) \le N(x, z) + N(y, z)$ for all $x, y, z \in X$.

The generalized 2-normed space is denoted by $(X, N(.,.))$.

Definition 2.2 *[4] A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(.,.))$ is called convergent if there exist $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, y) = 0$ then $\lim_{n\to\infty} N(x_n, y) = N(x, y)$ for all $y \in X$.*

Definition 2.3 *[4] A sequence $\{x_n\}$ in a generalized 2-normed space $(X, N(.,.))$ is called Cauchy sequence is there exist two lineary independent elements y and z in X such that $\{N(x_n, y)\}$ and $\{N(x_n, z)\}$ are real Cauchy sequences.*

Definition 2.4 *[4] A generalized 2-normed space $(X, N(.,.))$ is called generalized 2-Banach space is every Cauchy sequence is convergent.*

3 STABILITY RESULT IN GENERALIZED 2 - NORMED SPACE: DIRECT METHOD

In this section, the authors investigate the generalized Ulam - Hyers stability of the functional equation (1.7) in Generalized 2- normed space using direct method.

Now let us consider X be a generalized 2-normed space and Y be generalized 2-Banach space, respectively.

Theorem 3.1 *Let $j = \pm 1$. Let $g: X \to Y$ be a mapping for which there exist a function $\alpha, \delta: X^3 \to [0,\infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{2^{nj}} \alpha\left((2^{nj}x, s), (2^{nj}y, s), (2^{nj}z, s)\right) = 0 \tag{3.1}$$

such that the functional inequality

$$N\left(g(x) + g(y+z) - g(x+y) - g(z), s\right) \le \alpha((x, s), (y, s), (z, s)) \tag{3.2}$$

for all $x, y, z \in X$ and all $s \in X$. Then there exists a unique additive mapping $A: X \to Y$ satisfying the functional equation (3.7) and

$$N\big(g(x)-A(x),s\big)\leq\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\delta(2^{kj}x,s)}{2^{kj}} \tag{3.3}$$

where

$$\delta(2^{kj}x,s)=\alpha((2^{kj}x,s),(2^{kj}x,s),(0,s)) \tag{3.4}$$

for all $x\in X$ and all $s\in X$. The mapping $A(x)$ is defined by

$$\lim_{n\to\infty}N\left(\frac{g(2^{nj}x)}{2^{nj}},s\right)=N\big(A(x),s\big) \tag{3.5}$$

for all $x\in X$ and all $s\in X$.

Proof. Assume $j=1$. Replacing (x,y,z) by $(x,x,0)$ in (3.2), we get

$$N\left(g(x)-\frac{g(2x)}{2},s\right)\leq\frac{1}{2}\alpha\big((x,s),(x,s),(0,s)\big) \tag{3.6}$$

for all $x\in X$ and all $s\in X$. It follows from (3.6) that

$$N\left(g(x)-\frac{g(2x)}{2},s\right)\leq\frac{\delta((x,s))}{2} \tag{3.7}$$

where

$$\delta((x,s))=\alpha\big((x,s),(x,s),(0,s)\big)$$

for all $x\in X$ and all $s\in X$. Now replacing x by $2x$ and dividing by 2 in (3.7), we obtain

$$N\left(\frac{g(2x)}{2}-\frac{g(2^2x)}{2^2},s\right)\leq\frac{\delta(2x,s)}{2^2} \tag{3.8}$$

for all $x\in X$ and all $s\in X$. Using $(G4)$, it from (3.7) and (3.8), we have

$$\begin{aligned}N\left(g(x)-\frac{g(2^2x)}{2^2},s\right)&\leq N\left(g(x)-\frac{g(2x)}{2},s\right)+N\left(\frac{g(2x)}{2}-\frac{g(2^2x)}{2^2},s\right)\\&\leq\frac{1}{2}\left(\delta(x,s)+\frac{\delta(2x,s)}{2}\right)\end{aligned} \tag{3.9}$$

for all $x\in X$ and all $s\in X$. Proceeding further and using induction on a positive integer n, we get

$$N\left(g(x)-\frac{g(2^nx)}{2^n},s\right)\leq\frac{1}{2}\sum_{k=0}^{n-1}\frac{\delta(2^kx,s)}{2^k} \tag{3.10}$$

$$\le \frac{1}{2}\sum_{k=0}^{\infty}\frac{\delta(2^k x, s)}{2^k}$$

for all $x \in X$ and all $s \in X$. In order to prove the convergence of the sequence

$$\left\{\frac{g(2^n x)}{2^n}\right\},$$

replace x by $2^m x$ and dividing by 2^m in (3.10), for any $m, n > 0$, we deduce

$$N\left(\frac{g(2^m x)}{2^m} - \frac{g(2^{n+m} x)}{2^{(n+m)}}, s\right) = \frac{1}{2^m} N\left(g(2^m x) - \frac{g(2^n \cdot 2^m x)}{2^n}, s\right)$$

$$\le \frac{1}{2}\sum_{k=0}^{n-1}\frac{\delta(2^{k+m} x, s)}{2^{k+m}}$$

$$\le \frac{1}{2}\sum_{k=0}^{\infty}\frac{\delta(2^{k+m} x, s)}{2^{k+m}}$$

$$\to 0 \ as\ m \to \infty$$

for all $x \in X$ and all $s \in X$. Also

$$N\left(\frac{g(2^m x)}{2^m} - \frac{g(2^{n+m} x)}{2^{(n+m)}}, s_1\right) = \frac{1}{2^m} N\left(g(2^m x) - \frac{g(2^n \cdot 2^m x)}{2^n}, s_1\right)$$

$$\le \frac{1}{2}\sum_{k=0}^{n-1}\frac{\delta(2^{k+m} x, s_1)}{2^{k+m}}$$

$$\le \frac{1}{2}\sum_{k=0}^{\infty}\frac{\delta(2^{k+m} x, s_1)}{2^{k+m}}$$

$$\to 0 \ as\ m \to \infty$$

for all $x \in X$ and all $s_1 \in X$.

Hence there exists two linearly independent elements s and s_1 in X such that

$$\left\{N\left(\frac{g(2^n x)}{2^n}, s\right)\right\} \text{ and } \left\{N\left(\frac{g(2^n x)}{2^n}, s_1\right)\right\}$$

are real Cauchy sequences. Hence the sequence $\left\{\frac{g(2^n x)}{2^n}\right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $A : X \to Y$ such that

$$\lim_{n\to\infty} N\left(\frac{g(2^n x)}{2^n}, s\right) = N\big(A(x), s\big)\ \forall\ x \in X, s \in X$$

Letting $n \to \infty$ in (3.10) we see that (3.3) holds for all $x \in X$. To prove that A satisfies (1.7), replacing (x, y, z) by $(2^n x, 2^n y, 2^n z)$ and dividing by 2^n in (3.2), we obtain

$$\frac{1}{2^n} N(g(2^n x) + g(2^n (y+z)) - g(2^n (x+y)) - g(2^n z), s)$$
$$\leq \frac{1}{2^n} \alpha((2^n x, s), (2^n y, s), (2^n z, s))$$

for all $x, y, z \in X$ and all $s \in X$. Letting $n \to \infty$ in the above inequality using (3.7) and the definition of $A(x)$ and $(M1)$, we see that

$$A(x) + A(y+z) = A(x+y) + A(z).$$

Hence A satisfies (1.7) for all $x, y, z \in X$ and all $s \in X$. To prove that $A(x)$ is unique, let $B(x)$ be another additive mapping satisfying (1.7) and (3.3), then

$$N\left(A(x) - B(x), s\right) = \frac{1}{2^n} N\left(A(2^n x) - B(2^n x), s\right)$$
$$\leq \frac{1}{2^n}\left\{N\left(A(2^n x) - g(2^n x), s\right) + N\left(g(2^n x) - B(2^n x), s\right)\right\}$$
$$\leq \sum_{k=0}^{\infty} \frac{\delta(2^{k+n} x, s)}{2^{(k+n)}}$$
$$\to 0 \ \ as\ n \to \infty$$

for all $x \in X$ and all $s \in X$. Hence A is unique.

For $j = -1$, we can prove a similar stability result. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers [9], Ulam-TRassias [20] and Ulam-JRassias [19] stabilities of (1.7).

Corollary 3.2 *Let $g : X \to Y$ be a function and there exits real numbers λ and t such that*

$$N\left(g(x) + g(y+z) - g(x+y) - g(z), s\right)$$
$$\leq \begin{cases} \lambda, & \\ \lambda\left\{\|x,s\|^t + \|y,s\|^t + \|z,s\|^t\right\}, & t < 1 \quad or \quad t > 1; \\ \lambda\left\{\|x,s\|^t \|y,s\|^t \|z,s\|^t + \left\{\|x,s\|^{3t} + \|y,s\|^{3t} + \|z,s\|^{3t}\right\}\right\}, & t < \frac{1}{3} \quad or \quad t > \frac{1}{3}; \end{cases} \tag{3.11}$$

for all $x, y, z \in X$ and all $s \in X$. Then there exists a unique additive function $A : X \to Y$ such that

$$N\big(g(x)-A(x),s\big)\leq\begin{cases}\lambda,\\ \dfrac{2\lambda\,\|x,s\|^{t}}{|2-2^{t}|},\\ \dfrac{2\lambda\,\|x,s\|^{3t}}{|2-2^{3t}|},\end{cases}\tag{3.12}$$

for all $x\in X$ and all $s\in X$.

4 STABILITY RESULT IN GENERALIZED 2 - NORMED SPACE: FIXED POINT METHOD

In this section, the authors has proved the generalized Ulam - Hyers stability of Additive functional equation (1.7) in Generalized 2-normed spaces with the help of fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 4.1 *[14](The alternative of fixed point) Suppose that for a complete generalized metric space* (X,d) *and a strictly contractive mapping* $T:X\to X$ *with Lipschitz constant* L*. Then, for each given element* $x\in X$*, either*

(B_1) $\quad d(T^n x,T^{n+1}x)=\infty \quad \forall\ n\geq 0,$

or

(B_2) there exists a natural number n_0 such that:

(i) $d(T^n x,T^{n+1}x)<\infty$ for all $n\geq n_0$;

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y=\{y\in X: d(T^{n_0}x,y)<\infty\}$;

(iv) $d(y^*,y)\leq\dfrac{1}{1-L}\ d(y,Ty)$ for all $y\in Y$.

Hereafter through out this section, let us assume V be a vector space and B Banach space respectively.

Theorem 4.2 *Let* $g:V\to B$ *be a mapping for which there exist a function* $\alpha,\delta,\gamma:V^3\to[0,\infty)$ *with the condition*

$$\lim_{k\to\infty}\frac{1}{\mu_i^k}\alpha\big((\mu_i^k x,s),(\mu_i^k y,s),(\mu_i^k z,s)\big)=0\tag{4.1}$$

where

$$\mu_i = \begin{cases} 2, & i=0, \\ \frac{1}{2}, & i=1 \end{cases}$$

satisfying the functional inequality

$$N\big(g(x)+g(y+z)-g(x+y)-g(z),s\big) \le \alpha((x,s),(y,s),(z,s)) \tag{4.2}$$

for all $x,y,z \in V$ and all $s \in V$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x,s) = \delta\left(\frac{x}{2}, s\right),$$

has the property

$$\gamma(x,s) = L\,\mu_i\,\gamma\left(\frac{x}{\mu_i}, s\right) \tag{4.3}$$

for all $x \in V$ and all $s \in V$. Then there exists unique additive function $A: V \to B$ satisfying the functional equation (1.7) and

$$N\big(g(x)-A(x),s\big) \le \frac{L^{1-i}}{1-L}\gamma(x,s) \tag{4.4}$$

holds for all $x \in V$ and all $s \in V$.

Proof. Consider the set $X = \{p/p: V \to B,\ p(0)=0\}$ and introduce the generalized metric on X,

$$d(p,q) = \inf\{K \in (0,\infty) : N\big(p(x)-q(x),s\big) \le K\gamma(x,s), x \in V\}.$$

It is easy to see that (X,d) is complete.

Define $T: X \to X$ by

$$Tp(x) = \frac{1}{\mu_i} p(\mu_i x), \forall \quad x \in V.$$

Now $p,q \in X$,

$$d(p,q) \le K \Rightarrow N\big(p(x)-q(x),s\big) \le K\gamma(x,s), x \in V.$$

$$\Rightarrow N\left(\frac{1}{\mu_i}p(\mu_i x) - \frac{1}{\mu_i}q(\mu_i x), s\right) \le \frac{1}{\mu_i}K\gamma(\mu_i x, s), x \in V,$$

$$\Rightarrow N\left(\frac{1}{\mu_i}p(\mu_i x) - \frac{1}{\mu_i}q(\mu_i x), s\right) \le LK\gamma(x,s), x \in V,$$

$$\Rightarrow N\big(Tp(x)-Tq(x),s\big) \le LK\gamma(x,s), x \in V,$$

$$\Rightarrow d(Tp,Tq) \le LK.$$

This implies
$d(Tp,Tq) \le Ld(p,q),$

for all $p,q \in X$. i.e., T is a strictly contractive mapping on X with Lipschitz constant L.

From (3.7), we have

$$N\left(g(x)-\frac{g(2x)}{2}, s\right) \le \frac{\delta((x,s))}{2} \tag{4.5}$$

where
$\delta((x,s)) = \alpha\big((x,s),(x,s),(0,s)\big)$

for all $x \in V$ and all $s \in V$. Using (4.3) for the case $i = 0$, it reduces to

$$N\left(g(x)-\frac{1}{2}g(2x), s\right) \le \frac{1}{2}\gamma(x,s)$$

for all $x \in V$ and all $s \in V$.

i.e., $\quad d(g,Tg) \le \frac{1}{2} = L = L^{1-0} = L^{1-i} < \infty.$

Again replacing $x = \frac{x}{2}$ in (4.5), we get

$$N\left(2g\left(\frac{x}{2}\right)-g(x), s\right) \le \delta\left(\frac{x}{2}, s\right).$$

for all $x \in V$ and all $s \in V$. Using (4.3) for the case $i = 1$, it reduces to

$$N\left(2g\left(\frac{x}{2}\right)-g(x), s\right) \le \gamma(x,s)$$

for all $x \in V$ and all $s \in V$.

i.e., $\quad d(Tg,g) \le 1 = L^0 = L^{1-1} = L^{1-i} < \infty.$

In above cases, we arrive
$d(g,Tg) \le L^{1-i}$

Therefore $(B_2(i))$ holds.

By $(B_2(ii))$, it follows that there exists a fixed point A of T in X such that

$$N(A(x),s) = \lim_{k\to\infty} N\left(\frac{g(\mu_i^k x)}{\mu_i^k}, s\right) \qquad \forall\ x \in V,\ \ andall\ \ s \in V. \tag{4.6}$$

In order to prove $A: V \to B$ is Additive. Replacing (x,y,z) by $\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)$ in

(4.2) and dividing by μ_i^k, it follows from (4.1) and (4.6), A satisfies (4.7) for all $x, y, z \in V$ and all $s \in V$. i.e., A satisfies the functional equation (4.7).

By $(B_2(iii))$, A is the unique fixed point of T in the set $Y = \{g \in X : d(Tg, A) < \infty\}$, using the fixed point alternative result A is the unique function such that

$$N\left(g(x) - A(x), s\right) \leq K\gamma(x, s)$$

for all $x \in V$ and all $s \in V$ and $K > 0$. Finally by $(B_2(iv))$, we obtain

$$d(g, A) \leq \frac{1}{1-L} d(g, Tg)$$

this implies

$$d(g, A) \leq \frac{L^{1-i}}{1-L}.$$

Hence we conclude that

$$N\left(g(x) - A(x), s\right) \leq \frac{L^{1-i}}{1-L}\gamma(x, s).$$

for all $x \in V$ and all $s \in V$. This completes the proof of the theorem.

From Theorem 4.2, we obtain the following corollary concerning the Ulam-Hyers [9], Ulam-TRassias [20] and Ulam-JRassias [19] stabilities of (1.7).

Corollary 4.3 *Let $g : V \to B$ be a mapping and there exits real numbers λ and s such that*

$$N\left(g(x) + g(y+z) - g(x+y) - g(z), s\right)$$
$$\leq \begin{cases} (i) & \lambda, \\ (ii) & \lambda\left\{\|x,s\|^t + \|y,s\|^t + \|z,s\|^t\right\}, \qquad t < 1 \;\; or \;\; t > 1; \\ (iii) & \lambda\left\{\|x,s\|^t \|y,s\|^t \|z,s\|^t + \left\{\|x,s\|^{3t} + \|y,s\|^{3t} + \|z,s\|^{3t}\right\}\right\}, \quad t < \frac{1}{3} \;\; or \;\; t > \frac{1}{3}; \end{cases} \tag{4.7}$$

for all $x, y, z \in V$ and all $s \in V$, then there exists a additive function $A : V \to B$ such that

$$N\big(g(x)-A(x),s\big)\leq\begin{cases}(i) & \lambda,\\ (ii) & \dfrac{2\lambda\,\|x,s\|^{t}}{|2-2^{t}|},\\ (iii) & \dfrac{2\lambda\,\|x,s\|^{3t}}{|2-2^{3t}|},\end{cases} \tag{4.8}$$

for all $x\in V$ and all $s\in V$.

Proof. Setting

$$\alpha(x,y,z)=\begin{cases}\lambda,\\ \lambda\left\{\|x,s\|^{t}+\|y,s\|^{t}+\|z,s\|^{t}\right\},\\ \lambda\left\{\|x,s\|^{t}\|y,s\|^{t}\|z,s\|^{t}+(\|x,s\|^{3t}+\|y,s\|^{3t}+\|z,s\|^{3t})\right\}\end{cases}$$

for all $x,y,z\in V$ and all $s\in V$. Now

$$\frac{\alpha(\mu_i^k x,\mu_i^k y,\mu_i^k z)}{\mu_i^k}=\begin{cases}\dfrac{\lambda}{\mu_i^k},\\ \dfrac{\lambda}{\mu_i^k}\left\{\|\mu_i^k x,s\|^{t}+\|\mu_i^k y,s\|^{t}+\|\mu_i^k z,s\|^{t}\right\},\\ \dfrac{\lambda}{\mu_i^k}\{\|\mu_i^k x,s\|^{t}\|\mu_i^k y,s\|^{t}\|\mu_i^k z,s\|^{t}\left\{\|\mu_i^k x,s\|^{3t}+\|\mu_i^k y,s\|^{3t}+\|\mu_i^k z,s\|^{3t}\right\}\}\end{cases}$$

$$=\begin{cases}\to 0\ as\ \ k\to\infty,\\ \to 0\ as\ \ k\to\infty,\\ \to 0\ as\ \ k\to\infty.\end{cases}$$

i.e., (4.1) is holds. But we have

$$\gamma(x,s)=\delta\left(\frac{x}{2},s\right)=\alpha\left(\left(\frac{x}{2},s\right),\left(\frac{x}{2},s\right),(0,s)\right)=\begin{cases}\lambda,\\ \dfrac{2\lambda}{2^{t}}\|x,s\|^{t},\\ \dfrac{2\lambda}{2^{3t}}\|x,s\|^{3t},\end{cases}$$

Also,

$$\frac{1}{\mu_i}\gamma(\mu_i x,s)=\begin{cases}\dfrac{\lambda}{\mu_i},\\ \dfrac{2\lambda}{\mu_i\cdot 2^{t}}\|\mu_i x,s\|^{t},\\ \dfrac{2\lambda}{\mu_i\cdot 2^{3t}}\|\mu_i x,s\|^{3t}.\end{cases}=\begin{cases}\mu_i^{-1}\lambda,\\ \mu_i^{t-1}\dfrac{2\lambda}{2^{t}}\|x,s\|^{t},\\ \mu_i^{3t-1}\dfrac{2\lambda}{2^{3t}}\|x,s\|^{3t}.\end{cases}=\begin{cases}\mu_i^{-1}\gamma(x),\\ \mu_i^{t-1}\gamma(x),\\ \mu_i^{3t-1}\gamma(x).\end{cases}$$

Hence the inequality (4.3) holds either, $L = 2^{-1}$ for $t = 0$ if $i = 0$ and $L = 2$ for $t = 0$ if $i = 1$. Now from (4.4), we prove the following cases for condition (i).

Case:1 $L = 2^{-1}$ for $t = 0$ if $i = 0$

$$N\big(g(x) - A(x), s\big) \le \frac{L^{1-i}}{1-L}\gamma(x,s) = \frac{\left(2^{-1}\right)^{1-0}}{1-(2)^{-1}}\lambda = \lambda.$$

Case:2 $L = 2$ for $t = 0$ if $i = 1$

$$N\big(g(x) - A(x), s\big) \le \frac{L^{1-i}}{1-L}\gamma(x) = \frac{(2)^{1-1}}{1-2}\lambda = -\lambda.$$

Also, (4.3) holds either, $L = 2^{t-1}$ for $t < 1$ if $i = 0$ and $L = \frac{1}{2^{t-1}}$ for $t > 1$ if $i = 1$. Now from (4.4), we prove the following cases for condition (ii).

Case:1 $L = 2^{t-1}$ for $t < 1$ if $i = 0$

$$N\big(g(x) - A(x), s\big) \le \frac{L^{1-i}}{1-L}\gamma(x,s) = \frac{\left(2^{t-1}\right)^{1-0}}{1-2^{t-1}}\frac{2\lambda}{2^t}\|x,s\|^t = \frac{2^t}{2-2^t}\frac{2\lambda}{2^t}\|x,s\|^t = \frac{2\lambda\|x,s\|^t}{2-2^t}.$$

Case:2 $L = \frac{1}{2^{t-1}}$ for $t > 1$ if $i = 1$

$$N\big(g(x) - A(x), s\big) \le \frac{L^{1-i}}{1-L}\gamma(x,s) = \frac{\left(\frac{1}{2^{t-1}}\right)^{1-1}}{1-\frac{1}{2^{t-1}}}\frac{2\lambda}{2^t}\|x,s\|^t = \frac{2^t}{2^t-2}\frac{2\lambda}{2^t}\|x,s\|^t = \frac{2\lambda\|x,s\|^t}{2^t-2}.$$

Finally, (4.3) holds either, $L = 2^{3t-1}$ for $3t < 1$ if $i = 0$ and $L = \frac{1}{2^{3t-1}}$ for $3t > 1$ if $i = 1$. Now from (4.4), we prove the following cases for condition (iii).

Case:1 $L = 2^{3t-1}$ for $3t < 1$ if $i = 0$

$$N\big(g(x) - A(x), s\big) \le \frac{L^{1-i}}{1-L}\gamma(x,s) = \frac{\left(2^{3t-1}\right)^{1-0}}{1-2^{3t-1}}\frac{2\lambda}{2^{3t}}\|x,s\|^{3t} = \frac{2^{3t}}{2-2^{3t}}\frac{2\lambda}{2^{3t}}\|x,s\|^{3t} = \frac{2\lambda\|x,s\|^{3t}}{2-2^{3t}}.$$

Case:2 $L = \frac{1}{2^{3t-1}}$ for $3t > 1$ if $i = 1$

$$N\big(g(x)-A(x),s\big)\leq\frac{L^{1-i}}{1-L}\gamma(x,s)=\frac{\left(\frac{1}{2^{3t-1}}\right)^{1-1}}{1-\frac{1}{2^{3t-1}}}\frac{2\lambda}{2^{3t}}\|x,s\|^{3t}=\frac{2^{3t}}{2^{3t}-2}\frac{2\lambda}{2^{3t}}\|x,s\|^{3t}=\frac{2\lambda\|x\|^{3t}}{2^{3t}-2}.$$

Hence the proof of the corollary is complete.

References

[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ, Press, 1989.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64-66.

[3] M. Arunkumar, *Solution and Stability of Arun-Additive functional equations*, International Journal Mathematical Sciences and Engineering Applications, Vol 4, No. 3, (2010), 33-46.

[4] M. Arunkumar, G. Ganapathy, S. Murthy, *Stability of a functional equation having nth order solution in Generalized 2-Normed Spaces*, International Journal Mathematical Sciences and Engineering Applications, Vol. 5 No. IV (2011), 361-369.

[5] M. Arunkumar, S. Hema latha, C. Devi Shaymala Mary, *Functional equation originating from arithmetic Mean of consecutive terms of an arithmetic Progression are stable in banach space: Direct and fixed point method*, JP Journal of Mathematical Sciences, 3(1), (2012), 27-43.

[6] M. Arunkumar, P. Agilan, *Additive functional equation and inequality are Stable in Banach space and its applications*, Malaya Journal of Matematik (MJM), 1(1), (2013), 10-17.

[7] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[8] P. Gavruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184 (1994), 431-436.

[9] D.H. Hyers, *On the stability of the linear functional equation*, Proc.Nat.Acad.Sci.,U.S.A.,27 (1941) 222-224.

[10] D.H. Hyers, G. Isac,Th.M. Rassias, *Stability of functional equations in several variables*, Birkhauser, Basel, 1998.

[11] S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[12] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, 2009.

[13] D.O. Lee, *Hyers-Ulam stability of an addtiive type functional equation*, J. Appl. Math. and Computing, 13 (2003) no.1-2, 471-477.

[14] B.Margoils and J.B.Diaz, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. 126 74 (1968), 305-309.

[15] V.Radu , *The fixed point alternative and the stability of functional equations, in: Seminar on Fixed Point Theory Cluj-Napoca*, Vol. IV, 2003, in press.

[16] J.M. Rassias, *On approximately of approximately linear mappings by linear mappings*, J. Funct. Anal. USA, 46, (1982) 126-130.

[17] J.M. Rassias, *On approximately of approximately linear mappings by linear mappings*, Bull. Sc. Math, 108, (1984) 445-446.

[18] K. Ravi, M. Arunkumar, *On a n- dimensional additive Functional Equation with fixed point Alternative*, Proceedings of International Conference on Mathematical Sciences 2007.

[19] K. Ravi, M. Arunkumar and J.M. Rassias, *On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation*, International Journal of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.

[20] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc.Amer.Math. Soc., 72 (1978), 297-300.

[21] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions,Wiley, NewYork, 1964 (Chapter VI, Some Questions in Analysis: 1, Stability).