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Namias Fractional Hankel Transform in the Zemanian Space

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Abstract :

In this paper some properties of kernel of Namias fractional Hankel transform are proved and fractional Hankel transform is extended in the distributional generalized sense. Testing function space is defined. Analyticity, inversion theorem and uniqueness theorem for the generalized fractional Hankel transform are proved.

Keyword: Hankel transform, Fractional Hankel transform, Testing function space.

AMS Subject Classification: 46F12, 44.

I. Introduction :

Namias [5] introduced the concept of Hankel transform of fractional order by the method of eigen values. Kerr [3] also defined fractional Hankel transform which was studied by us in detail in [7, 8, 9]. The fractional Hankel transform have many applications in optics [2, 6], quantum mechanics [4], signal processing [6]. The fractional Hankel transform defined by Namias is,

$$H_v^{\propto}[f(x)](y) = \int_0^{\infty} K_{\propto}(x,y) f(x)dx \quad (1)$$

Where

$$K_{\propto}(x,y) = x\frac{e^{i(v+1)\left[\frac{\pi}{2}-\frac{\propto}{2}\right]}}{sin\frac{\propto}{2}}e^{-\left(\frac{ix^2}{2}+\frac{iy^2}{2}\right)cot\frac{\propto}{2}}J_v\left(\frac{xy}{sin\frac{\propto}{2}}\right) \quad (2)$$

$$=B_{v,\propto}xe^{-\frac{i}{2}(x^2+y^2)cot\frac{\propto}{2}}J_v\left(\frac{xy}{sin\frac{\propto}{2}}\right)$$

$$B_{v,\propto} = \frac{e^{i(v+1)\left[\frac{\pi}{2}-\frac{\propto}{2}\right]}}{sin\frac{\propto}{2}}$$

is the generalization of the conventional Hankel transform

$$H_v^{\propto}[f(x)] = \int_0^{\infty} f(x)J_v(xy)xdx \quad (3)$$

and the fractional Hankel transform defined by Kerr [3] is a generalization of Hankel transform defined by Zemanian [10]. It is interesting to develop some properties of above defined kernel (2) and to extend Namias fractional Hankel transform in the distributional generalized sense.

In this paper section II discusses some properties of kernel where as in section III we define testing function space. Section IV gives inversion theorem and section V uniqueness theorem, last section concludes the paper.

II. Properties of Kernel:

We prove the following properties of kernel of fractional Hankel transform.

1. $K_{\propto}(y,x) = \frac{y}{x}K_{\propto}(x,y)$
2. $K_{-\propto}(x,y) = K_{\propto}^{*}(x,y)$, where '$*$' denotes the conjugation
3. For $\propto = \pi$ the kernel coincides with the kernel of the Hankel transform given in (3)
4. $K_{\propto}(x,0) = 0 = K_{\propto}(0,y)$
5. $K_{\propto}(-x,y) = e^{-i\pi(v+1)}K_{\propto}(x,y)$
6. $\int_0^{\infty} K_{\propto}(x,y)\,K_{\beta}(y,z)dy = K_{\propto+\beta}(x,z)$

Proof: First five properties are simple to prove, hence we prove the last property.

(6) $\int_0^{\infty} K_{\propto}(x,y)\,K_{\beta}(y,z)dy = K_{\propto+\beta}(x,z)$

L.H.S. = $\int_0^{\infty} K_{\propto}(x,y)\,K_{\beta}(y,z)dy$

$$= B_{v,\propto}B_{v,\beta}e^{-\frac{i}{2}\left(x^2cot\frac{\alpha}{2}+z^2cot\frac{\beta}{2}\right)}x\int_0^{\infty} ye^{-\frac{i}{2}y^2\left(cot\frac{\alpha}{2}+cot\frac{\beta}{2}\right)}J_v\left(\frac{xy}{sin\frac{\alpha}{2}}\right)J_v\left(\frac{xy}{sin\frac{\beta}{2}}\right)dy \quad (4)$$

let first evaluate $\int_0^{\infty} ye^{-iAy^2}J_v(x_1y)J_v(z_1y)dy$,

where $A = \left(cot\frac{\alpha}{2} + cot\frac{\beta}{2}\right)$, $x_1 = \left(\frac{x}{sin\frac{\alpha}{2}}\right)$, $z_1 = \left(\frac{z}{sin\frac{\beta}{2}}\right)$

$$\int_0^{\infty} ye^{-iAy^2}J_v(x_1y)J_v(z_1y)dy = z_1^{-\frac{1}{2}}\left\{\int_0^{\infty}(z_1y)^{\frac{1}{2}}y^{\frac{1}{2}}J_v(x_1y)J_v(z_1y)cos(Ay^2)dy - i\int_0^{\infty}(z_1y)^{\frac{1}{2}}y^{\frac{1}{2}}J_v(x_1y)J_v(z_1y)sin(Ay^2)dy\right\}$$

Using results (26, 27) on page no. 57 of [1].

$$L.H.S. = \frac{x}{\left(\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}\right)} J_v\left(\frac{xz}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\left(\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}\right)}\right) e^{i\left(\frac{z^2\sin^2\frac{\alpha}{2}+x^2\sin^2\frac{\beta}{2}}{2\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}\left(\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}\right)}-(v+1)\frac{\pi}{2}\right)}$$

Equation (4) gives,

$$L.H.S. = B_{v,\alpha}B_{v,\beta}e^{-\frac{i}{2}\left(x^2\cot\frac{\alpha}{2}+z^2\cot\frac{\beta}{2}\right)} \frac{x}{\left(\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}\right)} J_v\left(\frac{xz}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\left(\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}\right)}\right) e^{i\left(\frac{z^2\sin^2\frac{\alpha}{2}+x^2\sin^2\frac{\beta}{2}}{2\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}\left(\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}\right)}-(v+1)\frac{\pi}{2}\right)}$$

after some straight forward steps we obtain,

$$\text{L. H. S.}= \frac{e^{i(v+1)\left(\frac{\pi}{2}-\frac{\alpha+\beta}{2}\right)}}{\sin\left(\frac{\alpha+\beta}{2}\right)} J_v\left(\frac{xz}{\sin\left(\frac{\alpha+\beta}{2}\right)}\right) x e^{-\frac{i}{2}(x^2+z^2)\cot\left(\frac{\alpha+\beta}{2}\right)}$$

$$= B_{v,\alpha+\beta}\, x e^{-\frac{i}{2}(x^2+z^2)\cot\left(\frac{\alpha+\beta}{2}\right)} J_v\left(\frac{xz}{\sin\left(\frac{\alpha+\beta}{2}\right)}\right)$$

$$= K_{\alpha+\beta}(x,z)$$

= R. H. S.

III The testing function Space E:

An infinitely differentiable complex valued function $\emptyset$ on R^n belongs to E(R^n) or E if for each compact set $B \subset S_a$, where $S_a = \{x\epsilon R^n, |x| \le a, a > 0\}$, $\gamma_{B,k}(\emptyset) = \sup_{x\epsilon B}|D^k\emptyset(x)| < \infty$, $\sup_{x\epsilon B}|D^k\emptyset(x)| < \infty$, k$\epsilon N^n$.

Clearly E is complete and so a Frechet space.

The fractional Hankel transform on E':

It is easily seen that for each $x\epsilon R^n$ and $0 < \alpha < 2\pi$ the function $K_\alpha(x,y)$ belongs to E as a function of x.

Hence the fractional Hankel transform of $f\epsilon E'$ can be defined by

$$H_v^\alpha[f(x)](y) = F(y) = \langle f(x), K_\alpha(x,y)\rangle \quad (5)$$

where

$$K_\alpha(x,y) = \frac{e^{i(v+1)\left[\frac{\pi}{2}-\frac{\alpha}{2}\right]}}{\sin\frac{\alpha}{2}} x e^{-\left(\frac{ix^2}{2}+\frac{iy^2}{2}\right)\cot\frac{\alpha}{2}} J_v\left(\frac{xy}{\sin\frac{\alpha}{2}}\right)$$

then the right hand side of (5) has a meaning as the application of $f\epsilon E'$ to $K_\alpha(x,y)\epsilon E$.

IV Inversion Theorem:

Let $f\epsilon E'(R), 0 \leq \propto \leq \pi$ and supp$f$ subset of $S_a$where $S_a = \{x\epsilon R^n, |x| \leq a, a > 0\}$ and let $F(y)$ be the generalized fractional Hankel transformation of $f$defined by

$$H_v^{\propto}[f(x)](y) = F(y) = \langle f(x), K_{\propto}(x,y)\rangle$$

Then for each $\emptyset\epsilon E$, we have

$$\langle f(x), \emptyset(x)\rangle = \langle \int_0^{\infty} \overline{K_{\propto}(x,y)}\, F(y)dy, \emptyset(x)\rangle$$

Where

$$\overline{K_{\propto}(x,y)} = \frac{e^{-i(v+1)\left[\frac{\pi}{2}-\frac{\propto}{2}\right]}}{\sin\frac{\propto}{2}} x e^{\left(\frac{ix^2}{2}+\frac{iy^2}{2}\right)\cot\frac{\propto}{2}} J_v\left(\frac{xy}{\sin\frac{\propto}{2}}\right)$$

Proof :- To prove the inversion theorem, we state the following lemmas to be used in the sequel.

Lemma 1:-
Let $H_v^{\propto}[f(x)](y) = F(y)$ for $0 < \alpha < \pi$ and supp S_a for $\emptyset(x)\epsilon E$

$$\psi(y) = \int_0^{\infty} \overline{K_{\propto}(x,y)}\, \emptyset(x)dx$$

Then for any fixed number r, $0 < r < \infty$

$$\int_0^r \psi(y)\langle f(\zeta), K_{\propto}(\zeta,y)\rangle d\tau = \langle f(\zeta), \int_0^r \psi(y) K_{\propto}(\zeta,y)\, d\tau\rangle \tag{6}$$

where $y = \sigma + i\tau\epsilon C^n$ and ζ is restricted to a compact subset of R.

Proof:- The case $\emptyset(x) = 0$ is trivial, let $\emptyset(x) \neq 0$,

It can be easily seen that, $\int_0^r \psi(y)\, K_{\propto}(\zeta,y)d\tau$, where $y = \sigma + i\tau$ is C^{∞} - function of ζ and it belongs to E. Hence the right hand side of (6) is meaningful.

To prove the equality, we construct the Riemann-sum for this integral and write

$$\int_0^r \psi(y)\langle f(\zeta), K_{\propto}(\zeta,y)\rangle d\tau = \lim_{m\to\infty} \langle f(\zeta), \sum_{n=0}^{m-1} K_{\propto}(\zeta, \sigma + i\tau_{n,m})\rangle \psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m}.$$

We show that the last summation converges in E to the integral on the right hand side of (6). Consider,

$$\gamma_{B,k}\left\{\sum_{n=0}^{m-1} K_{\propto}(\zeta, \sigma + i\tau_{n,m})\psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m} - \int_0^r \psi(y)\, K_{\propto}(\zeta,y)\right\} d\tau$$

$$= \sup_{\zeta\epsilon B}\left|\left\{\sum_{n=0}^{m-1} D_{\zeta}^k K_{\propto}(\zeta, \sigma + i\tau_{n,m})\psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m} - \int_0^r \psi(y)\, D_{\zeta}^k K_{\propto}(\zeta,y)d\tau\right\}\right|$$

Carrying the operator D_ζ^k within the integral and summation signs, which is easily justified.We get,

$$\lim_{m \to \infty} \sum_{n=0}^{m-1} D_\zeta^k K_\propto(\zeta, \sigma + i\tau_{n,m})\psi(\sigma + i\tau_{n,m})\Delta\tau_{n,m} = \int_0^r \psi(y)\, K_\propto(\zeta, y) d\tau, \forall\, \zeta \epsilon B.$$

It thus follows that for every m, the summation is a member of E and it converges in E to the integral on the right hand side of (6).

Hence the proof.

Lemma 2:-

For $\emptyset(x)\epsilon E$, set $\psi(y)$ as in Lemma 1 above for $y\epsilon C$, ζ is restricted to a compact subset of R then

$$M_r(\zeta) = \int_0^r K_\propto(\zeta, y) \int_0^\infty \overline{K_\propto(x, y)}\, \emptyset(x) dx d\tau \tag{7}$$

converges in E to $\psi(\zeta)$ as $r \to \infty$.

Proof:-

We shall show that $M_r(\zeta) \to \emptyset(\zeta)$ in E as $r \to \infty$. That is to show,

$$\gamma_{B,k}[M_r(\zeta) - \psi(\zeta)] = \sup_{\zeta \epsilon B}\{D_\zeta^k[M_r(\zeta) - \emptyset(\zeta)]\} \to 0 \text{ as } r \to \infty.$$

We note that for k=0,

$$\int_0^r K_\propto(\zeta, y) \left(\int_0^\infty \overline{K_\propto(x, y)}\, \emptyset(x) dx \right) d\tau = \emptyset(\zeta)$$

That is to say that $\lim_{r \to \infty} M_r(\zeta) = \emptyset(\zeta)$

Since the integrand is a C^∞- function of ζ and $\emptyset \in E$, we can repeatedly differentiate under the integral sign in (7) and the integrals are uniformly convergent. We have

$$\int_0^r D_\zeta^k K_\propto(\zeta, y) \left(\int_0^\infty \overline{K_\propto(x, y)}\, \emptyset(x) dx \right) d\tau = \emptyset(\zeta)$$

for all $\zeta \in B$.

Hence the claim.

Proof of inversion Theorem:

Let $\phi(x) \in E$. We shall show that

$$\langle \int_0^r \overline{K_\propto(x, y)}\, F(y) d\tau, \emptyset(x) \rangle \text{ tends to } \langle f(x), \emptyset(x) \rangle \text{ as } r \to \infty \tag{8}$$

From the analyticity of $F(y)$ on C and the fact that $\emptyset(x)$ has a compact support in R, it follows that the left side expression in (8) is merely a repeated integral with respect to x and y and the integral in (8) is a continuous function of x as the closed bounded domain of the integration.

Therefore we write (8) as,

$$\int_0^{\infty} \emptyset(x) \int_0^{r} \overline{K_{\propto}(x,y)}\, F(y) d\tau dx = \int_0^{r} \psi(y) \langle F(\zeta), K_{\propto}(\zeta,y)\rangle d\tau$$

Since $\emptyset(x)$ is of compact support, and the integrand is a continuous function of (x,y) the order of integration may be changed. The change in the order of integration is justified, where

$$\psi(y) = \int_0^{\infty} \overline{K_{\propto}(x,y)}\, \emptyset(x) dx$$

This yields

$$\int_0^{r} \psi(y) \langle F(\zeta), K_{\propto}(\zeta,y)\rangle d\tau = \langle F(\zeta), \int_0^{\infty} \psi(y) K_{\propto}(\zeta,y)\, d\tau\rangle \qquad (9)$$

again by lemma 2, RHS of equation (9) converges to $\langle f(\zeta), \emptyset(\zeta)\rangle$ as $r \to \infty$.

This completes the proof of the theorem.

V. Uniqueness Theorem :

If $H_{v}^{\propto}[f(x)](y) = F(y)$ and $H_{v}^{\propto}[g(x)](y) = G(y)$ for $0 < \propto \leq \frac{\pi}{2}$ suppf subset of S_a and suppg subset of S_a where $S_a = \{x\epsilon R, |x| \leq a, a > 0\}$,

if $F(y) = G(y)$

then *f* = *g* in the sense of equality in E'

Proof :- By inversion theorem

$$f - g = \int_0^{\infty} \overline{K_{\propto}(x,y)}\, [F(y) - G(y)] dy$$

= 0, as $F(y) = G(y)$

thus $f = g$ in E'

This proves uniqueness.

VI Conclusion:-

We present some properties of kernel. The inversion theorem for the generalized fractional Hankel transform with two lemmas is given. Uniqueness theorem is also given. Further we plan to prove operation transform formulae and some more interesting properties of this transform.

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