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Convexity preserving rational cubic spline with multiple shape parameters

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Abstract

The new weighted C^2 rational cubic spline has been constructed by using two kinds of rational cubic spline has been constructed by using two kinds of rational cubic spline with linear denominator. The necessary and sufficient conditions that constrain the interpolating curve to be convex (or concave) in the interpolating interval are derived. Also the error bound of this interpolation is discussed.

AMS subject classification:
Keywords: Rational cubic spline, shape parameters, interpolation, convexity, approximation.

1. Introduction

Now a days the area of research is concentrated on computer graphics and engineering design. Many authors are working on the modification of the desired curve by using shape parameters [3, 5, 6, 8, 11]. The rational cubic spline [1, 2, 4, 6, 7, 8] is a powerful tool for designing curves, surfaces and for geometric shapes. Rational cubic spline functions of lower degree are numerically simple, suitable and fundamental of all rational space curves. M. Sarfraz [10] has worked on rational cubic spline interpolation with shape control. Duan [11] has constructed a weighted rational cubic spline interpolation using two kinds of rational cubic spline with quadratic denominator. We have extended the

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work by Duan. In this paper, using the idea of Duan, we have constructed rational cubic spline with linear denominator and with multiple shape parameters. We have discussed the convexity of such rational cubic spline.

The paper is arranged as follows. In Section 2, weighted rational cubic spline is constructed using two kinds of rational cubic spline with linear denominator and multiple shape parameters. C^2 continuity of the rational cubic spline is discussed in section 3. In section 4, convexity in the interpolating interval or subinterval is derived. Section 5 is about the approximation properties of such rational cubic spline.

2. Weighted rational cubic spline interpolation

Let $t_0 < t_1 < \cdots < t_n < t_{n+1}$ are the knots f_i and d_i $i = 0, 1, 2, 3, \ldots, n, n+1$ are the function values and derivatives defined at the knots. Let $\theta_i = (t - t_i)/h_i$ for $t \in [t_i, t_{i+1}]$ and let α_i and β_i be positive parameters.

Now consider,

$$p^*(t) = \frac{p_i^*(t)}{q_i^*(t)} \tag{1}$$

where,

$$p_i^*(t) = (1-\theta)^2 \alpha_i A_i^* + \theta(1-\theta)^2 B_i^* + \theta^2(1-\theta) C_i^* + \theta^2 \beta_i D_i^*$$
$$q_i^*(t) = (1-\theta)\alpha_i + \theta\beta_i$$

which satisfying the following interpolatory conditions

$$p_i^*(t) = f_i \text{ and } p_i^{*'}(t) = d_i, \tag{2}$$

for $i = 0, 1, 2, \ldots, n, n+1$. Thus we have,

$$A_i^* = f_i, \ D_i^* = f_{i+1}$$
$$B_i^* = (\alpha_i + \beta_i) f_i + \alpha_i h_i d_i \text{ and}$$
$$C_i^* = (\alpha_i + \beta_i) f_{i+1} - \beta_i h_i d_{i+1}$$

similarly we denote another rational cubic spline with linear denominator by the following

$$p_*(t) = \frac{p_{i,*}(t)}{q_{i,*}(t)} \tag{3}$$

which satisfies the following interpolatory conditions

$$p_*(t_i) = f_i \ \text{ and } \ p_*'(t_i) = \Delta_i \tag{4}$$

where,

$$\Delta_i = \frac{f_{i+1} - f_i}{h_i} \quad \text{for } i = 0, 1, 2, \ldots, n$$

and

$$p_{i,*} = (1-\theta)^2 \alpha_i A_{i,*} + \theta(1-\theta)^2 B_{i,*} + \theta^2(1-\theta) C_{i,*} + \theta^2 \beta_i D_{i,*}$$

$$q_{i,*} = (1-\theta)\alpha_i + \theta\beta_i$$

after some calculations

$$A_{i,*} = f_i, \; D_{i,*} = f_{i+1}$$

$$B_{i,*} = \alpha_i f_{i+1} + \beta_i f_i \; \text{ and}$$

$$C_{i,*} = (\alpha_i + \beta_i) f_{i+1} - \beta_i h_i \Delta_{i+1}$$

It is called the rational cubic spline based on the function values.

The weighted rational cubic spline will be constructed by using the two kinds of rational interpolation described above. Let

$$p(t) = \lambda p^*(t) + (1-\lambda) p_*(t) \tag{5}$$

$\lambda \in [0, 1]$ where λ is known as weighted coefficient and $t \in [t_i, t_{i+1}]$. Therefore $p(t)$ can be written as

$$p(t) = \frac{p_i(t)}{q_i(t)}$$

$i = 0, 1, 2, \ldots, n.$

$$p_i(t) = (1-\theta)^2 \alpha_i A_i + \theta(1-\theta)^2 B_i + \theta^2(1-\theta) C_i + \theta^2 \beta_i D_i$$

$$q_i(t) = (1-\theta)\alpha_i + \theta\beta_i$$

Thus the rational cubic spline with weight coefficient is satisfying the following conditions

$$p(t_i) = f_i \; \text{ and}$$
$$p'(t_i) = \lambda_i d_i + (1-\lambda)\Delta_i$$

After some calculation we have

$$A_i = f_i \text{ and } D_i = f_{i+1}$$

$$B_i = (\beta_i + \lambda\alpha_i) f_i + (1-\lambda)\alpha_i f_{i+1} + \lambda\alpha_i h_i d_i \text{ and}$$

$$C_i = (\alpha_i + \beta_i) f_{i+1} + h_i \beta_i (\lambda(\Delta_{i+1} - d_{i+1}) - \Delta_{i+1})$$

for $i = 0, 1, 2, \ldots, n$.

It has been observed that $p(t)$ is a C^1-continuous, piecewise, rational cubic spline called the weighted rational cubic spline with linear denominator, as linear combination of the interpolations p^* and p_*. Note that p^* and p_* are defined in $[t_0, t_{n+1}]$ and $[t_0, t_n]$ respectively thus $p(t)$ is defined in $[t_0, t_n]$.

3. C^2 rational cubic spline

Further more this weighted rational spline could even be C^2 in the interpolating interval $[t_0, t_n]$. We apply the following C^2 rational cubic spline interpolant on (5)

$$p''(t_i+) = p''(t_i-), i = 1, 2, \ldots, n-1,$$

Thus we have the system of linear equations:

$$\begin{aligned} & h_{i-1}\beta_{i-1}(\alpha_i+\beta_i)\lambda(\Delta_i - d_i) + \beta_i(\Delta_i - \Delta_{i+1} + \lambda(\Delta_{i+1} - d_{i+1})) \\ &= h_i\alpha_i(\alpha_{i-1}+\beta_{i-1})(2\Delta_i - \Delta_{i-1} - \lambda(\Delta_i - d_i)) + \lambda\alpha_{i-1}(d_{i-1} - \Delta_{i-1}) \end{aligned} \tag{6}$$

$i = 1, 2, \ldots, n-1$.

Thus by selecting the values of α_{i-1}, β_{i-1} and β_i we get α_i. Where,

$$\alpha_i = \frac{\lambda\alpha_{i-1}(d_{i-1} - \Delta_{i-1}) - \beta_i(\Delta_i - \Delta_{i+1} + \lambda(\Delta_{i+1} - d_{i+1})) - h_{i-1}\beta_{i-1}\beta_i\lambda(\Delta_i - d_i)}{h_{i-1}\beta_{i-1}\lambda(\Delta_i - d_i) - h_i(\alpha_{i-1}+\beta_{i-1})(2\Delta_i - \Delta_{i-1} - \lambda(\Delta_i - d_i))}$$

then $p(t) \in c^2(t_0, t_n)$.

4. Convexity control of the rational cubic spline

To get the condition for the interpolation to keep convexity in the interpolating interval we shall consider the condition for the second order derivative to remain positive or negative in the interpolating interval. Assume the knots are equally spaced differentiate rational cubic spline twice the interpolant (5) in $[t_i, t_{i+1}]$, we have the following form

$$p''(t) = (h_i^2((1-\theta)\alpha_i + \theta\beta_i)^3)^{-1}Q(\theta) \tag{7}$$

where

$$\begin{aligned} Q(\theta) = {} & ((1-\theta)\alpha_i + \theta\beta_i)^2(2\alpha_i f_i + (-4+6\theta)B_i \\ & + (2-6\theta)C_i + 2\beta_i f_{i+1}) - 2((1-\theta)\alpha_i + \theta\beta_i)(-\alpha_i+\beta_i)(-2\alpha_i f_i(1-\theta) \\ & + (1-4\theta+3\theta^2)B_i + (2\theta - 3\theta^2)C_i + 2\theta\beta_i f_{i+1}) \\ & + 2(-\alpha_i+\beta_i)^2(\alpha_i f_i(1-\theta)^2 + \theta(1-\theta)^2B_i + \theta^2(1-\theta)C_i + \theta^2\beta_i f_{i+1}) \end{aligned} \tag{8}$$

Here we find that the denominator of $p''(t)$ is positive in $[t_i, t_{i+1}]$. It is easy to prove that $Q(\theta)$ is the cubic polynomial in the variable θ, let

$$Q(\theta) = A\theta^3 + B\theta^2 + C\theta + D$$

in which

$$A = 2(\alpha_i - \beta_i)^2(\beta_i(l - k\lambda) - m\lambda\alpha_i)$$
$$B = 6\alpha_i(\beta_i - \alpha_i)(\beta_i(l - k\lambda) - m\lambda\alpha_i)$$
$$C = 6\alpha_i^2(\beta_i(l - k\lambda) - m\lambda\alpha_i)$$
$$D = 2\alpha_i^2\beta_i\lambda m - 2\alpha_i^2(\beta_i(l - k\lambda) - m\lambda\alpha_i)$$

where

$$m = f_{i+1} - f_i - h_i d_i,$$
$$k = f_{i+2} - f_{i+1} - h_i d_{i+1},$$
$$l = f_{i+2} + f_i - 2f_{i+1}$$

Since the denominator of $p''(t)$ is positive, shown by (5), $p''(t) \geq 0$ is equivalent to $Q(\theta) \geq 0, \theta \in [0, 1]$. Let $\theta = \frac{s}{s+1}$, $Q(\theta)$ is equivalent to $U(s) = \alpha s^3 + \beta s^2 + \gamma s + \delta \geq 0, s \geq 0$, Where,

$$\alpha = A + B + C + D = 2\beta_i^2(l - k\lambda)(\alpha_i + \beta_i) - 2m\lambda\alpha_i\beta_i^2$$
$$\beta = B + 2C + 3D = 6\alpha_i\beta_i^2(l - k\lambda)$$
$$\gamma = C + 3D = 6\alpha_i^2\beta_i m\lambda$$
$$\delta = D = -2(l - k\lambda)\alpha_i^2\beta_i + 2\alpha_i^2 m\lambda(\alpha_i + \beta_i)$$

Using the sufficient and necessary condition for the interpolation function $p(t)$ to be convex on $[t_i, t_{i+1}]$ is the positive parameters α_i, β_i satisfy either

$$\alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta \geq 0, or$$

$$m \geq 0, l - k\lambda \geq 0, \frac{m\lambda(\alpha_i + \beta_i)}{\beta_i} \geq l - k\lambda \geq \frac{m\lambda\alpha_i}{\alpha_i + \beta_i} \tag{9}$$

Thus we have proved the following theorems.

Theorem 4.1. Given $(t_i, f_i, d_i), i = 0, 1, 2, \ldots, n, n + 1$, the sufficient and necessary condition for the interpolation defined by (5) to be convex on $[t_i, t_{i+1}]$ is that the given data and the positive parameters α_i, β_i and λ satisfy both

$$m \geq 0, l - k\lambda \geq 0, \frac{m\lambda(\alpha_i + \beta_i)}{\beta_i} \geq l - k\lambda \geq \frac{m\lambda\alpha_i}{\alpha_i + \beta_i}$$

Theorem 4.2. For the equally spaced knots, if the function being interpolated, $f(t)$, is convex in the interpolating interval $[t_0, t_{n+1}]$, then there must exist positive parameters α_i, β_i and λ which satisfy (10) to keep the interpolating curve to be convex.

5. Approximation properties of the weighted interpolation

To find the error estimation we consider that the given ion $f(t) \in C^2$ and $p(t)$ is the interpolating function of $f(t)$ in $[t_i, t_{i+1}]$ for $i = 0, 1, \ldots, n+1$, consider the case that the knots are equally spaced, namely, $hi = h = \frac{t_n - t_0}{n}$ for all $i = 1, 2, \ldots, n$, using the Peano-Kernel Theorem [9] gives the following

$$R[F] = f(t) - p(t) = \int f^2(\tau) R_t[(t-\tau)_+] d\tau, t \in [t_i, t_{i+1}],$$

where,

$$R_t[(t-\tau)_+] = \begin{cases} (t-\tau) - \dfrac{(\theta(1-\theta)^2(1-\lambda)\alpha_i + \theta^2\beta_i + \theta^2(1-\theta)(\alpha_i+\beta_i))(t_{i+1}-\tau) - \theta^2(1-\theta)h_i\beta_i}{(1-\theta)\alpha_i + \beta_i} \\ t_i < \tau < t \\ -\dfrac{(\theta(1-\theta)^2(1-\lambda)\alpha_i + \theta^2\beta_i + \theta^2(1-\theta)(\alpha_i+\beta_i))(t_{i+1}-\tau) - \theta^2(1-\theta)h_i\beta_i}{(1-\theta)\alpha_i + \beta_i} \\ t < \tau < t_{i+1} \\ \dfrac{\theta^2(1-\theta)(1-\lambda)\beta_i(t_{i+2}-\tau)}{(1-\theta)\alpha_i + \beta_i} \\ t_{i+1} < \tau < t_{i+2} \end{cases}$$

$$= \begin{cases} p(\tau), t_i < \tau < t \\ q(\tau), t < \tau < t_{i+1} \\ r(\tau), t_{i+1} < \tau < t_{i+2} \end{cases}$$

Consider when $\lambda \leq 1$ then

$$\|R[f]\| = \|f(t) - P(t)\| \leq \|f^{(2)}(t)\| \left[\int_{t_i}^{t} |p(\tau)| d\tau + \int_{t}^{t_{i+1}} |q(\tau)| d\tau + \int_{t_{i+1}}^{t+2} |r(\tau)| d\tau \right] \tag{10}$$

Next, we calculate the following terms in (11)

$$\int_{t_{i+1}}^{t} |r(\tau)| d\tau = \frac{\theta^2(1-\theta)h_i\beta_i(\lambda(\frac{t_{i+2}-\tau}{h_i}) - \frac{t_{i+2}-\tau}{h_i})}{(1-\theta)\alpha_i + \beta_i}$$

We observe that for

$$q(t) = -\frac{\theta(1-\theta)^2 h_i((1-\theta)(1-\lambda)\alpha_i + \theta(\alpha_i+\beta_i))}{(1-\theta)\alpha_i + \beta_i} \leq 0$$

and at

$$\tau = t_{i+1} q(t_{i+1}) = \frac{\theta^2(1-\theta)h_i\beta_i}{\alpha_i(1-\theta) + \theta\beta_i} \geq 0$$

It is easy to see that the root of $q(\tau)$ is

$$\tau^* = t_{i+1} - \frac{\theta(1-\theta)\beta_i h_i}{(1-\theta)((1-(1-\theta)\lambda)(\alpha_i+\theta\beta_i))}$$

Thus,

$$\int_t^{t_{i+1}} |q(\tau)| d\tau = \int_t^{\tau^*} -q(\tau) d\tau + \int_{\tau^*}^{t+1} q(\tau) d\tau$$

Similarly, since

$$p(t) = q(t) \leq 0, p(t_i) = \frac{\theta(1-\theta)^2\lambda\alpha_i}{(1-\theta)\alpha_i + \beta_i} \geq 0$$

$$\tau^* = t_{i+1} - \frac{h(\alpha_i + \theta\beta_i)}{\alpha_i + \theta\beta_i + \theta\lambda\alpha_i}$$

so that,

$$\int_t^{t_{i+1}} |p(\tau)| d\tau = \int_t^{\tau^*} -p(\tau) d\tau + \int_{\tau^*}^{t+1} p(\tau) d\tau$$

from the above calculations it can be shown that

$$\|R[f]\| = \|f(t) - P(t)\| \leq \|f^2(t)\| h^2 w(\theta, \alpha_i, \beta_i, \lambda) \tag{11}$$

where $w(\theta, \alpha_i, \beta_i, \lambda)$ is a constant depending upon $\theta, \alpha_i, \beta_i, \lambda$

6. Conclusion

In this paper, a weighted rational cubic spline with linear denominator was constructed. It was C^2 continuous in interpolating interval.The necessary and sufficient conditions was derived for the convexity of such spline function which is useful in the field of engineering.

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