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Existence of Smooth Homomorphism from a Fuchsian Group to a Molecular Symmetric Group

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Abstract

In this paper we consider a molecule (the cyclohexane molecule C_6H_{12}) and find the group of symmetries of the molecule $C_6H_{12},$ which is nothing but $D_2,$ a point grouph. Then we find a set of necessary and sufficient conditions for existence of a smooth Homomorphism from a Fuchsian group to this point group.

Keywords: Point group, Symmetries, Fuchsian group, Smooth quotient, Riemann surface, Automorphism group and Genus

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Introduction

It is observed that the study of symmetries one of the most appealing application of group theory. A symmetry of a figure is a rigid motion which leaves the figure unchanged i.e. a map $m: P \to P$ from a plane to itself is said to be rigid motion or an isometry if it preserves the distance between the molecules, such that $d(p,q) = d\big(m(p), m(q)\big) \forall p, q \in P$ where d is the distance between the points. If a rigid motion carries a subset F of the plane P to itself then we call it symmetry of F. It is observed that set of all symmetries of F always forms a group under the operation of functions, called group of symmetries of F $[Artin\ M]$.

It is found that every finite group is isometric to the automorphism group of some compact Riemann surface of genus (≥ 2) $[Burnside\ W]$.

The automorphisms (biholomorphic self-transformations) of a compact Riemann surface S of genus (≥ 2) form a finite group whose order can not exceed $84(g-1)$. The maximum bound is called Hurwitz bound and it is attained for infinitely many values of g, the least being 3 $[Macbeath\ A.M]$. The problem of finding minimum genus for various sub-class of finite groups has been the theme of many research papers during last few decades $[Chetia\ BP\&\ Patra\ K', Chutia\ C, Gromadzki\ G\ \&Maclanchlan\ C]$

In this paper we consider a molecule C_6H_{12} and then determine the group of symmetries of this molecule. Considering this point group we determine a set of necessary and sufficient condition for existence of smooth homomorphism from a Fuchsian group Γ to the point group associated with C_6H_{12}. Following these conditions we shall determine the upper bound for the order of the point group D_2 acting on compact Riemann surface of genus g which will appear in a separate paper.

Preliminaries: The theory of Fuchsian group is intimately related to the theory of Riemann surface automorphism groups. A Fuchsian group Γ is an infinite group having presentation of the form

$$\langle\, a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots . a_k; b_1, c_1, b_2, c_2 \ldots\ldots\ldots . . b_\gamma, c_\gamma\,; a_1^{m_1} = a_2^{m_2} = \cdots\ldots\ldots . . = a_k^{m_k} = \prod_{i=1}^{k} a_i \prod_{j=1}^{\gamma}[b_j, c_j] = 1\,\rangle \ldots\ldots \quad .(2.1)$$

Where $[b_j, c_j] = b_j^{-1}c_j^{-1}b_jc_j$ and $a_1, a_2, a_3, \ldots\ldots\ldots\ldots\ldots . a_k$ are of finite order generators of order $m_1, m_2, m_3, \ldots\ldots\ldots\ldots\ldots . m_k$ respectively and $b_1, c_1, b_2, c_2 \ldots\ldots\ldots . . b_\gamma, c_\gamma$ are finite order generators of Γ, and the measure of Γ is given by $\delta(\,\Gamma\,) = 2\gamma - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right) > 0$... . (2.2)

Such a Fuchsian is also denoted by $\Delta(\gamma;\ m_1, m_2, , \ldots\ldots\ldots\ldots\ldots . m_k)$ the non negative integer γ is called the genus Γ and the integer $m_i(\geq 2)$ are called the periods of Γ.

If $\gamma = 0$ then Γ has the signature of the form $\Gamma = \Delta(m_1, m_2, , \ldots\ldots\ldots\ldots\ldots . m_k)$ and if there is no finite order element except identity in Γ then $\Gamma = (\gamma\,; -)$, called a surface group.

It is known that if Γ_1 is a subgroup of Γ of finite index then Γ_1 is a Fuchsian group and

$$[\,\Gamma : \Gamma_1\,] = \frac{\delta(\Gamma_1)}{\delta(\Gamma)} \ldots\ldots \quad .(2.3)\ [Macbeath\ A.M].$$

A homomorphism φ from a Fuchsian group Γ to a finite group G is called smooth if the kernel is a surface subgroup of Γ. The factor group $\Gamma/\ker\varphi \cong \varphi(\Gamma)$ is called a smooth quotient.

Another remarkable result is that a finite group G is representable as an automorphism group of compact Riemann surface of genus g if and only if there is a smooth epimorphism φ from a Fuchsian group Γ to G such that $\ker\varphi$ has genus g. $[Macbeath\ A.M]$ i.e. if and only if $G \cong \Gamma/K$ where K is surface group of genus g.

Following these remarkable results in this paper we find a set of necessary and sufficient conditions on periods and genus of Fuchsian group Γ for which we have a smooth epimorphism from Γ to the point group generated by all the symmetries of the molecule C_6H_{12}.

A molecule has symmetry means that certain parts of it can be interchanged with others without altering either the identity or the orientation of the molecules $[Elton\, F.A.\&Wilkinson]$. The interchangeable parts are said to be equivalent to one another by symmetry.

Symmetry operations are geometrically defined ways of interchanging equivalent parts of a molecule. These are of four types as given below:

i. Simple rotation about an axis passing through the molecule by an angle $\frac{2\pi}{n}$. This operation is called proper rotation and is symbolised by 'C_n'.

ii. Reflection of all atoms through a planes which passes through the molecule. This operation is called reflection and is symbolised by 'σ'.

iii. Reflection of all atoms through a point in the molecule.This operation is called inversion and is symbolised by ' i '.

iv. Rotation about an axis passing through the molecule by $2\pi/n$ and then reflection through a plane perpendicular to this axis is called improper rotation and is symbolised by ' S_n '.

As already mentioned that the set of all symmetries of a molecule forms a group called group of symmetries or a point group. In this study we consider the molecule C_6H_{12} (cyclohexene), the organised molecule abide by the point group D_2 with symmetry elements E, the identity and three mutually perpendicular C_2axes i.e. $C_2(x)$, $C_2(y)$, $C_2(z)$. It is observed that the group can be represented by

$G = D_2 = \langle\, x, y \,/\, x^2 = y^2 = 1, xy = yx \,\rangle$.. (2.4)

Where 1= E, $x = C_2(x)$, $y = C_2(y)$, $xy = C_2(z)$.

Finally we establish a set of necessary and sufficient conditions for existence of smooth homomorphism from Fuchsian group Γ to the group of symmetries of the molecule C_6H_{12} as given below :

Main Result

In this section we find a smooth epimorphism from Γ to D_2.

Theorem : There is a smooth epimorphism φ from the Fuchsian group $\Gamma=\langle\, a_1, a_2, \dots\dots\dots . a_k; b_1, c_1, b_2, c_2 \dots\dots\dots .. b_\gamma, c_\gamma\,; a_1^{m_1} = a_2^{m_2} = \cdots \dots\dots .. = a_k^{m_k} = \prod_{i=1}^{k} a_i \prod_{j=1}^{\gamma}[b_j, c_j] = 1 \,\rangle$ with $\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{k}\left(1 - \frac{1}{m_i}\right) > 0$ to the symmetry group of molecule C_6H_{12} represented by $D_2 = \langle\, x, y \,/\, x^2 = y^2 = 1, xy = yx \,\rangle$ if and only if

1. If $k = 0$ i.e. $\Gamma = (\gamma\,; -)$, a surface group then $\gamma \geq 2$.
2. If $k \neq 0$ then $m_i = 2 and\ k \geq 2$. Moreover,

i. $\gamma \geq 0 \; if \; k \geq 5$
ii. $\gamma \geq 1 \; if \; k = 2, 3, 4$

Proof: Let $\varphi : \Gamma \to D_2$ be a smooth epimorphism then from (2, 1) it follows that $\gamma \geq 2$ when $k = 0$. If $k \neq 0$ then clearly $m_i = 2$ because φ preserves the periods of Γ. It is seen that if $k = 1$ then $\varphi(a_1) \prod_{j=1}^{\gamma} [\varphi(b_j), \varphi(c_j)]$=1 because φ is a smooth epimorphism.

But $\prod_{j=1}^{\gamma} [\varphi(b_j), \varphi(c_j)]$=1

$\Longrightarrow \varphi(a_1) = 1$, a contradiction. Thus $k \neq 1$ i.e. $k \geq 2$.

More over since φ is smooth, there exist compact Riemann surface of genus $g\ (\geq 2)$ on which D_2 acts as an automorphism group. Then by Riemann- Hurwitz formula (2.3) we have,

$$\frac{2(g-1)}{4} = 2(\gamma - 1) + \sum_{i=1}^{k} (1 - \frac{1}{m_i}), m_i = 2$$

$$\Rightarrow \frac{(g-1)}{2} = 2(\gamma - 1) + k(\frac{1}{2})$$

$$\Rightarrow g = 1 + 4(\gamma - 1) + k$$

$$\Rightarrow g = -3 + 4\gamma + k$$

Thus $\gamma \geq 0 \; if \; k \geq 5 \; and \; \gamma \geq 1 \; if \; k = 2, 3, 4 \; as \; g \geq 2$

This completes the proof.

Next we shall show that conditions are sufficient.

1.When $k = 0$ and $\gamma \geq 2$ we construct $\varphi : \Gamma \to D_2$ by

$\varphi(b_1) = x = \varphi(c_1)$

$\varphi(b_2) = y = \varphi(c_2)$

$\varphi(b_j) = 1 = \varphi(c_j) \; \forall \; j \geq 2$

Then clearly φ is a homomorphism and $ker\varphi$ being a subgroup of a surface group is a surface group. Hence φ is a smooth epimorphism.

2. (i) when $\gamma \geq 0$ and $k \geq 5$ we define a function $\varphi : \Gamma \to D_2$ by

$\varphi(a_1) = x = \varphi(a_2)$

$\varphi(a_i) = y \; for \; i = 3, 4, \dots \dots \dots \dots k \; when \; k \; is \; even$ and

$\varphi(b_j) = 1 = \varphi(c_j) \; ; j = 1, 2, \dots \dots \dots \dots ..$

Then φ is a smooth epimorphism as

$\prod_{i=1}^{k} \varphi(a_i) \prod_{j=1}^{\gamma} \left[\varphi(b_j), \varphi(c_j)\right] = 1$

If k is odd then let us define φ as

$\varphi(a_1) = y, \varphi(a_2) = x, \varphi(x_i) = xy \; for \; i = 3, \dots \dots . k$

$\varphi(b_j) = 1 = \varphi(c_j) \; \forall \; j = 1, 2, \dots \dots \dots \dots \gamma$

Then also φ is a smooth epimorphism.

2.(ii) when $\gamma \geq 1$ and k=2, 3, 4 let us construct
(a) $\varphi(a_1) = xy, \varphi(a_2) = x, \varphi(a_3) = y$, when k=3
$\varphi(b_1) = x = \varphi(c_1)$
$\varphi(b_j) = 1 = \varphi(c_j)\,; j = 2, 3, \dots \dots \dots \dots .. \gamma$

(b) $\varphi(a_i) = x, i = 1, 2, \dots \dots \dots k, k = 2, 4$
$\varphi(b_1) = y = \varphi(c_1)$
$\varphi(b_j) = 1 = \varphi(c_j)\,;\; j = 2, 3, \dots \dots \dots \dots . \gamma$

Then in both cases $\prod_{i=1}^{k} \varphi(a_i) \prod_{j=1}^{\gamma} \left[\varphi(b_j)_{,} \varphi(c_j)\right] = 1$

Implies that φ is a smooth epimorphis
This completes the theorem.
Thus we can conclude as follows:

Theorem: The group of symmetries 'G' of the molecule C_6H_{12} can be acted as a group of automorphism of some Riemann surface S of genus $g(\geq 2)$ under the following conditions :
There is a smooth epimorphism $\varphi : \Gamma \to G$ where
Γ and G are given by (2.1) and(2.4) respectively, iff

(I) If $k = 0$ i.e. $\Gamma = \Delta(\gamma; -)$, a surface group then $\gamma \geq 2$
(II)If $k \neq 0$ $then\ m_i = 2\ and\ k \geq 2$. Moreover,
(i) $\gamma \geq 0\ if\ k \geq 5$
(ii) $\gamma \geq 1\ if\ k = 2, 3, 4$

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