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Difference Cordial Labeling of Graphs

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Abstract

In this paper, we introduce a new notion called difference cordial labeling. Let G be a (p, q) graph. Let $f: V\ (G)\ \rightarrow \{1, 2, \ldots, p\}$ be a function. For each edge uv, assign the label $|f(u) - f\ (v)\ |$. f is called a difference cordial labeling if f is a one to one map and $\left|e_f(0) - e_f\ (1)\right| \leq 1$ where $e_f\ (1)$ and $e_f\ (0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph.

Keywords: Path, Cycle, Complete graph, Complete bipartite graph, Star, Helm.

Introduction

We consider finite, undirected and simple graphs only. Let $G = (V, E)$ be (p, q) graph. The cardinality of the vertex set and edge set respectively is called the order and size of a graph. Cahit introduced the concept of cordial labeling in the year 1987 in [1]. Let f be a function from $V\ (G)$ to $\{0, 1\}$ and for each edge xy, assign the label $|f(x) - f\ (y)\ |$. f is called a cordial labeling if the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1 and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. k-Product cordial labeling [4] has been introduced by R. Ponraj, M. Sivakumar and M. Sundaram. Motivated by these works we define here a new notion called difference cordial labeling of a graph. Terms and definitions not defined here are used in the sense of Harary [3].

Difference Cordial Labeling

Definition 2.1:
Let G be a (p, q) graph. Let f be a map from $V\ (G)$ to $\{1, 2, \dots, p\}$. For each edge uv, assign the label $|f(u) - f\ (v)\ |$. f is called difference cordial labeling if f is $1-1$ and $|e_f(0) - e_f\ (1)\ | \leq 1$ where $e_f\ (1)$ and $e_f\ (0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph.

Example: The following is a simple example of a difference cordial graph

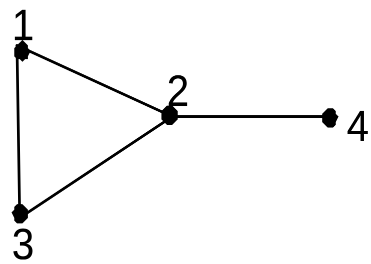

Figure (i)

Theorem 2.2: Every graph is a subgraph of a connected difference cordial graph.

Proof: Let G be a given (p, q) graph. Let $N = \frac{p\ (p-1)}{2} - 2p + 2$. Consider the complete graph K_p and the cycle C_N. Let $V(K_p) = \{u_1, u_2, \dots, u_p\}$ and C_N be the cycle $v_1 v_2 \dots v_N v_1$. We construct the super graph G^* of G as follows: Let $V(G^*) = V\ (K_p)\ \cup V\ (C_N)$ and $E(G^*) = E\ (K_p)\ \cup E\ (C_N)\ \cup \{u_p v_1\}$. Clearly G is a sub graph of G^*. Assign the label i to u_i $(1 \leq i \leq p)$ and $p + j$ to v_j $(1 \leq j \leq N)$. Therefore, $e_f(0) = N + p$ and $e_f(1) = N + p - 1$. Hence G^* is a difference cordial graph. ■

Theorem 2.3: If G is a (p, q) difference cordial graph, then $q \leq 2p - 1$.

Proof: Let f be a difference cordial labeling of G. Obviously, $e_f(1) \leq p - 1$. This implies $e_f(0) \geq q - p + 1$ $\rightarrow$. (1)

Case (i)
: $e_f(0) = e_f(1) + 1$.

From (1) , $q \leq e_f(0) + p - 1 = e_f(1) + 1 + p - 1 \leq 2p - 1$.
Therefore, $q \leq 2p - 1 \rightarrow (2)$.

Case (ii) : $e_f(1) = e_f(0) + 1$.
From (1) , $q \leq e_f(0) + p - 1 \leq e_f(0) + p - 1 \leq 2p - 2$.
Therefore, $q \leq 2p - 2 \rightarrow (3)$.

Case (ii) : $e_f(1) = e_f(0) + 1$.
From (1) , $q \leq e_f(0) + p - 1 = e_f(1) - 1 + p - 1 \leq 2p - 3$.
Therefore, $q \leq 2p - 3 \rightarrow (4)$.
From (2) , (3) and (4) , $q \leq 2p - 1$. ■

Theorem 2.4: If G is a r-regular graph with $r \geq 4$ then G is not difference cordial.

Proof: Let G be a (p, q) graph. Suppose G is difference cordial. Then by theorem 2.3, $q \leq 2p - 1$. This implies $q \leq \frac{4q}{r} - 1$. Hence $q \leq q - 1$. This is impossible. ■

Theorem 2.5: Any Path is a difference cordial graph.

Proof: Let P_n be the path $u_1 u_2 \dots u_n$. The following table (i) gives the difference cordial labeling of $P_n, n \leq 8$.

Table (i)

e	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
1	1							
2	1	2						
3	1	3	2					
4	1	2	4	3				
5	1	2	4	3	5			
6	1	2	3	5	4	6		
7	1	2	3	5	7	6	4	
8	1	2	3	4	6	8	7	5

Assume $n > 8$. Define a map $f: V(P_n) \rightarrow \{1, 2, \dots, n\}$ as follows:

Case (i) : $n \equiv 0 \ (mod\ 4)$.
Define

$$f(u_i) = i \quad 1 \leq i \leq \frac{n+2}{2}$$

$$f\left(u_{\frac{n+2}{2}+i}\right) = \frac{n+2}{2} + 2i \quad 1 \leq i \leq \frac{n}{4} - 1$$

$$f\left(u_{\frac{3n}{4}+i}\right) = \frac{n+2}{2} + 2i - 1 \quad 1 \leq i \leq \frac{n}{4}$$

Case (ii) : $n \equiv 1\ (mod\ 4)$.

Define

$$f(u_i) = i \quad 1 \leq i \leq \frac{n+1}{2}$$
$$f\left(u_{\frac{n+1}{2}+i}\right) = \frac{n+1}{2} + 2i \quad 1 \leq i \leq \frac{n-1}{4}$$
$$f\left(u_{\frac{3n+1}{4}+i}\right) = \frac{n+1}{2} + 2i - 1 \quad 1 \leq i \leq \frac{n-1}{4}$$

Case (iii) : $n \equiv 2\ (mod\ 4)$.

Define

$$f(u_i) = i \quad 1 \leq i \leq \frac{n+2}{2}$$
$$f\left(u_{\frac{n+2}{2}+i}\right) = \frac{n+2}{2} + 2i \quad 1 \leq i \leq \frac{n-2}{4}$$
$$f\left(u_{\frac{3n+2}{4}+i}\right) = \frac{n+2}{2} + 2i - 1 \quad 1 \leq i \leq \frac{n-2}{4}$$

Case (iv) : $n \equiv 3\ (mod\ 4)$.

Define

$$f(u_i) = i \quad 1 \leq i \leq \frac{n+1}{2}$$
$$f\left(u_{\frac{n+1}{2}+i}\right) = \frac{n+1}{2} + 2i \quad 1 \leq i \leq \frac{n+1}{4} - 1$$
$$f\left(u_{\frac{3n-1}{4}+i}\right) = \frac{n+1}{2} + 2i - 1 \quad 1 \leq i \leq \frac{n+1}{4}$$

The following table (ii) proves that *f* is a difference cordial labeling.

Table (ii)

Nature of *n*	$e_f\ (0)$	$e_f\ (1)$
$n \equiv 0\ (mod\ 2)$	$\frac{n-2}{2}$	$\frac{n}{2}$
$n \equiv 1\ (mod\ 2)$	$\frac{n-1}{2}$	$\frac{n-1}{2}$

■

Corollary 2.6: Any Cycle is a difference cordial graph.

Proof: The function f in theorem 2.5 is also a difference cordial labeling of the cycle C_n: $u_1u_2 \ldots u_nu_1$. ∎

Theorem 2.7: The Star $K_{1,n}$ is difference cordial iff $n \leq 5$.

Proof: Let $V(K_{1,n}) = \{u, u_i: 1 \leq i \leq n\}$, $E(K_{1,n}) = \{uu_i: 1 \leq i \leq n\}$. Table (iii) shows that the star $K_{1,n}, n \leq 5$ is difference cordial.

Table (iii)

n	u	u_1	u_2	u_3	u_4	u_5
1	1	2				
2	1	2	3			
3	1	2	3	4		
4	2	1	3	4	5	
5	2	1	3	4	5	6

Assume $n > 5$. Suppose f is a difference cordial labeling of $K_{1,n}, n > 5$. Without loss of generality assume that $f(u) = x$. To get the edge label 1, the only possibility is that $f(u_i) = x - 1, f(u_j) = x + 1$ for some i, j. This implies $e_f(1) \leq 2$. $e_f(0) - e_f(1) \geq n - 2 - 2 > 1$, a contradiction. ∎

Theorem 2.8: K_n is difference cordial iff $n \leq 4$.

Proof: Suppose K_n is difference cordial. Then $\frac{n\,(n-1)}{2} \leq 2n - 1$. This implies $n \leq 4$. K_1, K_2 are difference cordial by theorem 2.5. Using corollary 2.6, K_3 is difference cordial. A difference cordial labeling of K_4 is given in figure (ii) .

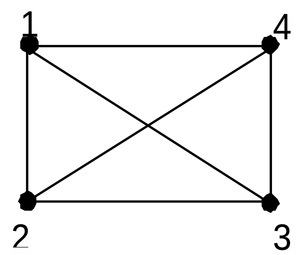

Figure (ii)

∎

Now we look into the complete bipartite graph $K_{m,n}$.

Theorem 2.9: If $m \geq 4$ and $n \geq 4$, then $K_{m,n}$ is not difference cordial.

Proof: Suppose $K_{m,n}$ is difference cordial. By theorem 2.3, $mn \leq 2(m+n) - 1$. This implies $mn - 2m - 2n + 1 \leq 0$, a contradiction to $m \geq 4$ and $n \geq 4$. ∎

Theorem 2.10: $K_{2,n}$ is difference cordial iff $n \leq 4$.

Proof: Let $V(K_{2,n}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_i: 1 \leq i \leq n\}$. $K_{1,2}, K_{2,2}$ are difference cordial by theorem 2.5 and corollary 2.6 respectively. A difference cordial labeling of $K_{2,3}$ and $K_{2,4}$ are given in figure (iii) .

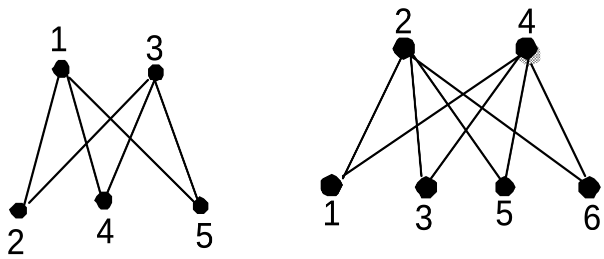

Figure (iii)

Now, we assume $n \geq 5$. Suppose f is a difference cordial labeling. Let $f(u_1) = r_1, f(u_2) = r_2$. Then $K_{2,n}$ has at most 4 edges with label 1. The maximum value is attained if the vertices in the set V_2 receive the labels $r_1 - 1, r_1 + 1, r_2 - 1, r_2 + 1$. Therefore $e_f(1) \leq 4, e_f(0) \geq 2n - 4$. Hence $e_f(0) - e_f(1) \geq 2n - 8 > 2$, a contradiction. ∎

Theorem 2.11: $K_{3,n}$ is difference cordial iff $n \leq 4$.

Proof: $K_{3,1}, K_{3,2}$ are difference cordial graphs by theorem 2.7 and 2.10. A difference cordial labeling of $K_{3,3}, K_{3,4}$ are given in figure (iv)

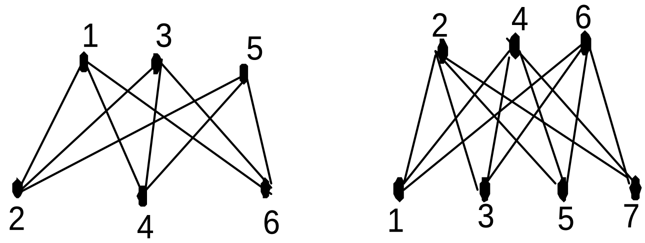

Figure (iv)

For any injective map f on $V(K_{3,5})$, $e_f(1) \leq 6$. Therefore, $K_{3,5}$ is not difference cordial. Assume $n \geq 6$. Suppose $K_{3,n}$ is difference cordial, then by theorem 2.3, $3n \leq 2(n+3)-1$, a contradiction. ■

Now we investigate the difference cordial labeling behavior of bistar $B_{m,n}$.

Theorem 2.12: If $m+n \geq 9$ then $B_{m,n}$ is not difference cordial.

Proof: Let $V(B_{m,n}) = \{u, v, u_i, v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(B_{m,n}) = \{uu_i, vv_j, uv : 1 \leq i \leq m, 1 \leq j \leq n\}$. Assume $f(u) = x$ and $f(v) = y$.

Case (i) : $y \neq x-1$ and $y \neq x+1$.

To get the edge label 1, u_i and u_j must receive the labels $x-1, x+1$ respectively for some i, j and v_i, v_j must receive the labels $y-1, y+1$ respectively for some i, j. Hence $e_f(1) \leq 4$.

Case (ii) : $y = x-1$ or $x+1$.

Obviously, in this case $e_f(1) \leq 3$. Thus by case (i) , (ii) , $e_f(1) \leq 4$. Therefore, $e_f(0) \geq q-4 \geq m+n-3$. Then $e_f(0)-e_f(1) \geq m+n-3-4 \geq 2$, a contradiction. ■

Theorem 2.13: $B_{1,n}$ is difference cordial iff $n \leq 5$.

Proof: Let $V(B_{1,n}) = \{u, v, u_1, v_i : 1 \leq i \leq n\}$ and $E(B_{1,n}) = \{uu_1, vv_i, uv : 1 \leq i \leq n\}$.

Case (i) : $n \leq 5$. $B_{1,1}$ is difference cordial by theorem 2.4. A difference cordial labeling of $B_{1,2}$ is in figure (v)

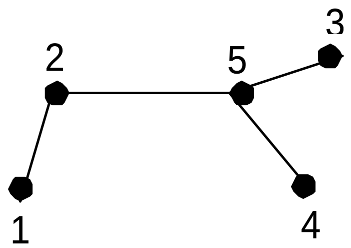

Figure (v)

For $3 \leq n \leq 5$, define $f(u) = 2, f(v) = 4, f(u_1) = 1, f(v_1) = 3, f(v_i) = 3 + i, 2 \leq i \leq n$. Clearly this f is a difference cordial labeling.

Case (ii) : $n > 5$.
When $n \geq 8$, the result follows from theorem 2.12. When $n = 6\ or\ 7$, $e_f\ (1)\ \leq 3$. Therefore, $e_f(0) \geq 5\ or\ 6$ according as $n = 6\ or\ 7$. This is a contradiction. ■

Theorem 2.14: $B_{2,n}$ is difference cordial iff $n \leq 6$.

Proof: Let $V(B_{2,n}) = \{u, v, u_1, u_2, v_i : 1 \leq i \leq n\}$ and $E(B_{m,n}) = \{uu_1, uu_2, vv_i, uv : 1 \leq i \leq n\}$.

Case (i) : $n = 2, 3$.
When $n = 2$, define $f(u) = 4, f(v) = 2, f(u_1) = 6, f(u_2) = 5, f(v_1) = 1, f(v_2) = 3$. When $n = 3$, define $f(u) = 5, f(v) = 2, f(u_1) = 6, f(u_2) = 7, f(v_1) = 1, f(v_2) = 3, f(v_3) = 4$. Clearly, f is a difference cordial labeling.

Case (ii) : $n = 4, 5, 6$.
Define $f(v) = 2, f(v_1) = 1, f(v_i) = 1 + i, 2 \leq i \leq n, f(u_1) = n + 2, f(u) = n + 3, f(u_2) = n + 4$. In this case, $e_f(0) = n - 1$ and $e_f(1) = 4$. Therefore, f is a difference cordial labeling.

Case (iii) : $n \geq 7$.
Proof follows from theorem 2.12. ■

Theorem 2.15: $B_{3,n}$ is difference cordial iff $n \leq 5$.

Proof: Let $V(B_{3,n}) = \{u, v, u_i, v_j : 1 \leq i \leq 3, 1 \leq j \leq n\}$ and $E(B_{m,n}) = \{uu_i, vv_j, uv : 1 \leq i \leq 3, 1 \leq j \leq n\}$. $B_{3,1}, B_{3,2}$ are difference cordial by theorems 2.13 and 2.14 respectively. For $3 \leq n \leq 5$, define $f(v) = 2, f(v_1) = 1, f(v_i) = 1 + i, 2 \leq i \leq n, f(u_1) = n + 2, f(u) = n + 3, f(u_2) = n + 4, f(u_3) = n + 5$. In this case, $e_f(0) = n$ and $e_f(1) = 4$. Therefore, f is a difference cordial labeling. For $n \geq 6$, the result follows from theorem 2.12. ■

Remark 2.16: $B_{4,4}$ is difference cordial.

Theorem 2.17: The wheel W_n is difference cordial.

Proof: Let $W_n = C_n + K_1$ where C_n is the cycle $u_1u_2 \dots u_nu_1$ and $V(K_1) = \{u\}$. Define a map $f: V(W_n) \to \{1, 2, \dots n + 1\}$ by $f(u) = 1, f(v_i) = i + 1, 1 \leq i \leq n$. Then $e_f(0) = n, e_f(1) = n$. ■

Corollary 2.18: The fan F_n is difference cordial for all n.

Proof: Let $F_n = P_n + K_1$ where P_n is the path $u_1u_2 \dots u_n$ and $V(K_1) = \{u\}$. The function f given in theorem 2.17 is also a difference cordial labeling since $e_f(0) =$

$n-1, e_f(1) = n$. ■

The gear graph G_n is obtained from the wheel W_n by adding a vertex between every pair of adjacent vertices of the cycle C_n.

Theorem 2.19: The gear graph G_n is difference cordial.

Proof: Let the vertex set and edge set of the wheel W_n be defined as in theorem 2.17. Let $V(G_n) = V(W_n) \cup \{v_i : 1 \leq i \leq n\}$ and $E(G_n) = E(W_n) \cup \{u_i v_i, v_j u_{j+1} : 1 \leq i \leq n, 1 \leq j \leq n\} - E(C_n)$. Define $f: V(G_n) \to \{1, 2 \ldots 2n+1\}$ as follows:

Case (i) : *n* is even.

$f(u_i) = 2i - 1 \; 1 \leq i \leq \frac{3n+2}{4} \; if\; n \equiv 2 \;(mod\; 4)$

$1 \leq i \leq \frac{3n}{4} \; if\; n \equiv 0 \;(mod\; 4)$

$f(u_{n-i+1}) = \frac{3n}{2} + 3 + i \; 1 \leq i \leq \frac{n-2}{4} \; if\; n \equiv 2 \;(mod\; 4)$

$1 \leq i \leq \frac{n}{4} \; if\; n \equiv 0 \;(mod\; 4)$

$f(v_i) = 2i \; 1 \leq i \leq \frac{3n-2}{4} \; i\,f\; n \equiv 2 \;(mod\; 4)$

$1 \leq i \leq \frac{3n-4}{4} \; if\; n \equiv 0 \;(mod\; 4)$

$f(v_{n-i+1}) = \frac{3n}{2} + i \; 1 \leq i \leq \frac{n+2}{4} \; if\; n \equiv 2 \;(mod\; 4)$

$1 \leq i \leq \frac{n+4}{4} \; if\; n \equiv 0 \;(mod\; 4)$

Case (ii) : *n* is odd.

$f(u_i) = 2i - 1 \; 1 \leq i \leq \frac{3n+1}{4} \; if\; n \equiv 1 \;(mod\; 4)$

$1 \leq i \leq \frac{3n+3}{4} \; if\; n \equiv 3 \;(mod\; 4)$

$$f(u_{n-i+1}) = \begin{cases} \frac{3n-1}{2} + 3 + i \; 1 \leq i \leq \frac{n-1}{4} \; if\; n \equiv 1 \;(mod\; 4) \\ \frac{3n+1}{2} + 3 + i \; 1 \leq i \leq \frac{n-3}{4} \; if\; n \equiv 1 \;(mod\; 4) \end{cases}$$

$f(v_i) = 2i \; 1 \leq i \leq \frac{3n+1}{4} \; if\; n \equiv 1 \;(mod\; 4)$

$1 \leq i \leq \frac{3n-1}{4} \; if\; n \equiv 3 \;(mod\; 4)$

$f(v_{n-i+1}) = \frac{3n+3}{2} + i - 1 \; 1 \leq i \leq \frac{n-1}{4} \; if\; n \equiv 2 \;(mod\; 4)$

$1 \leq i \leq \frac{n+1}{4} \; if\; n \equiv 0 \;(mod\; 4)$

Finally we define $f(u) = 2n + 1$. Table (iv) establishes that f is a difference cordial labeling.

Table (iv)

Values of n	e_f (0)	e_f (1)
$n \equiv 0\ (mod\ 2)$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1\ (mod\ 2)$	$\frac{3n-1}{2}$	$\frac{3n+1}{2}$

■

We now look into helms and webs. The helm H_n is obtained from a wheel W_n by attaching a pendent edge at each vertex of the cycle C_n. Koh et al. [2] define a web graph. A web graph is obtained by joining the pendent points of the helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. Yang [2] has extended the notion of a web by iterating the process of adding pendent vertices and joining them to form a cycle and then adding pendent point to the new cycle. $W(t,n)$ is the generalized web with t cycles C_n. Note that $W(1,n)$ is the helm and $W(2,n)$ is the web.

Theorem 2.20: All webs are difference cordial.

Proof: Let $C_n^{(i)}$ be the cycle $u_1^i u_2^i \dots u_n^i u_1^i$. Let

$$V(W(t,n)) = \bigcup_{i=1}^{t} V\left(C_n^{(i)} \right) \cup \{u\} \cup \{v_i : 1 \le i \le n\}$$

And

$$E(W(t,n)) = \bigcup_{i=1}^{t} E\left(C_n^{(i)} \right) \cup \{uu_j^1 : 1 \le j \le n\} \cup \{u_j^n v_j : 1 \le j \le n\} \cup \{u_j^i u_j^{i+1} : 1 \le j \le n, 1 \le i \le t-1\}.$$

Define a map $f: V(W(t,n)) \to \{1, 2 \dots nt + n + 1\}$ as follows:

$f(u_{2i-1}^t) = 4i - 2$ $1 \le i \le \frac{n}{2}$ if n is even

$1 \le i \le \frac{n+1}{2}$ if n is odd

$f(u_{2i}^t) = 4i - 1$ $1 \le i \le \frac{n}{2}$ if n is even

$1 \le i \le \frac{n-1}{2}$ if n is odd

$f(v_{2i-1}^t) = 4i - 3$ $1 \le i \le \frac{n}{2}$ if n is even

$1 \le i \le \frac{n+1}{2}$ if n is odd

$f(v_{2i}^t) = 4i$ $1 \le i \le \frac{n}{2}$ if n is even

$1 \le i \le \frac{n-1}{2}$ if n is odd

$f\left(u_j^{t-j}\right) = (j + 1) + t + i$ $1 \le j \le t - 1, 1 \le i \le n - t.$

$f(u_{n-i}^{t-j}) = (j+1)n + j - i\ 1 \le j \le t-1, 0 \le i \le j-1.$
$f(u) = nt + n + 1$. From the following table (v) , f yields a difference cordial labeling of the webs $W(t, n)$

Table (v)

Nature of n	$e_f\ (0)$	$e_f\ (1)$
$n \equiv 0\ (mod\ 2)$	$\frac{n\ (2t+1)}{2}$	$\frac{n\ (2t+1)}{2}$
$n \equiv 1\ (mod\ 2)$	$\frac{n\ (2t+1)\ -1}{2}$	$\frac{n\ (2t+1)\ +1}{2}$

■

A web graph $W(3,5)$ with a difference cordial labeling is given in figure (vi)

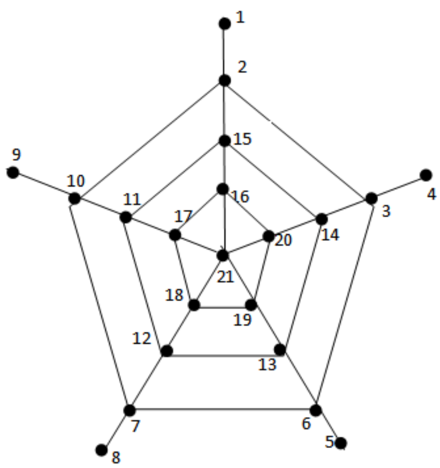

Figure (vi)

Corollary 2.21: The helm H_n is difference cordial for all n.

Proof: Since $H_n \cong W\ (1, n)$, the proof follows from theorem 2.20. ■

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