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Fractional Brownian Motion and Predictability Index in Financial Market

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Abstract

A predictability index for volatility in financial market is proposed. Market volatility of a particular day is suggested to be represented by a 4–vector consisting of opening value, highest value, lowest value and closing value of the volatility index (VIX) of that day. Regarding the stochastic market dynamics following fractional Brownian motion, the predictability is quantified using fractal dimension analysis of the time series for each of these volatility values. The predictability index is calculated for each of India VIX with underlying NIFTY options prices and CBOE with underlying S&P 500 stock index options prices. These are shown to follow fractional Brownian motion.

Keywords: Brownian motion, Fractal Brownian motion, volatility, Hurst exponent, Fractal dimension, predictability index.

1. Fractional Brownian motion

A *discrete Brownian motion* (BM) is a real–valued stochastic process $t_i \to B(t_i)$ over discrete time valuest_i, $i = 0,1,2, \ldots$ such that: (i) $B(t_0) = 0$ (ii) for $0 \leq i < j \leq k < l$ the increments $B(t_j) - B(t_i)$ and $B(t_l) - B(t_k)$ are independent random variables and (iii) for $0 < i < j$ the increments $B(t_j) - B(t_i)$ are normal random variables with mean zero and variance $\sim(t_j - t_i)$. Taking BM as the source of randomness, the celebrated Black–Scholes- Merton theory (1973) develops option pricing formula involving stochastic calculus which is based on integration with respect to the (Brownian motion), the underlying assumption being that the market follows a Brownian path.

A *discrete fractional Brownian motion* (fBM) is a stochastic process $t_i \to B_H(t_i)$ such that the increments $B_H(t_i) - B_H(t_j)$ have normal distributions with mean zero and variance $E\left(\left[B_H(t_i) - B_H(t_j)\right]^2\right) \sim \left(t_i - t_j\right)^{2H}$, E denoting the ensemble average and H is the Hurst exponent satisfying $0 < H < 1$.When $H = 0.5$ it becomes a BM. Thus fBM includes BM. Mandelbrot and van Ness (1968) suggested the use of fBM as a source of randomness for the financial market [2].

Starting with Rogers (1997), there is an ongoing debate on proper usage off BM in option pricing theory especially due to its inability to incorporate no-arbitrage pricing. However as shown in [5], arbitrage can be made to disappear by assuming that market participants cannot react instantaneously. Since fBM incorporates serial correlation which the observable market values seem to exhibit [5, 7], there is a considerable interest in using fBM in financial modeling. The process $t_i \to B_H(t_i)$ is known to satisfy the following [5]: For all i and j

(i) $E(B_H(t_i)) = 0$ (ii) $E\left(B_H(t_i)B_H(t_j)\right) = \frac{1}{2}\left[|t_i|^{2H} + |t_j|^{2H} - |t_i - t_j|^{2H}\right]$.

The increments $\Delta B_H(t_i, t_j) = B_H(t_i) - B_H(t_j)$satisfy for all i and j, $E\left(\Delta B_H(t_i, t_j)\right) = 0$ and$E\left(\left(\Delta B_H(t_i, t_j)\right)^2\right) = |t_i - t_j|^{2H}$. Except for $H = \frac{1}{2}$, the increments of fBM are not independent, since the covariance of increments $\Delta B_H(t_i, t_j)$ and $\Delta B_H(t_j, t_k)$ is

$$E\left(\Delta B_H(t_i, t_j)\Delta B_H(t_j, t_k)\right) = \frac{1}{2}\left[{t_i}^{2H} - {t_j}^{2H} - \left(t_i - t_j\right)^{2H}\right]$$

A fBM is known to exhibit self similarity; the resulting curve is a fractal whose fractal dimension is given by $D = 2 - H$.

2. Predictability index for volatility

If the fractal dimension $D$for the time series is 1.5, then the process is unpredictable, in view of the independence of time series increments in BM. If it decreases to 1, the process becomes more and more predictable as it exhibits "persistence". If the fractal dimension increases from 1.5 to 2, the process exhibits "anti – persistence". In either case, predictability arises. Motivated by climate predictability index in Atmospheric Science [3, 4], here we suggest predictability index for stock market parameters. As an example, we consider the volatility of the market. We define volatility of the market as a quadruple vector $V = (V_O, V_H, V_L, V_C)$, where $V_O =$ volatility at the opening of the day, V_H= highest volatility during the day, V_L= lowest volatility during the day and V_C= volatility at the closing of the day. Assuming each of these parameters V_O, V_H, V_L, V_C following fBM, the predictability index vector of volatility is defined to be the quadruple $PI_V = (PI_O, PI_H, PI_L, PI_C)$,where $PI_O = 2|D_O - 1.5|$, $PI_H = 2|D_H - 1.5|$, $PI_L = 2|D_L - 1.5|$, and $PI_C = 2|D_C - 1.5|$. HereD_O, D_H, D_L, D_C are fractal dimensions of the time series V_O, V_H, V_L, V_C respectively.

Appropriate norms can be used to measure ‘size’ of the predictability index, viz,
$\|PI_V\|_S = |PI_O| + |PI_H| + |PI_L| + |PI_C|$;
$\|PI_V\|_M = max(PI_O, PI_H, PI_L, PI_C)$;
$\|PI_V\|_E = (|PI_O|^2 + |PI_H|^2 + |PI_L|^2 + |PI_C|^2)^{\frac{1}{2}}$.

On the other hand, these norms of the volatility vector (V_O, V_H, V_L, V_C) give
$\|V\|_S = |V_O| + |V_H| + |V_L| + |V_C|$;
$\|V\|_M = max(V_O, V_H, V_L, V_C)$
$\|V\|_E = (|V_O|^2 + |V_H|^2 + |V_L|^2 + |V_C|^2)^{\frac{1}{2}}$

resulting into three time series.

We apply these ideas to the available data on CBOE VIX and India VIX. The Hurst exponent is computed following rescaled range $\frac{R}{S}$ analysis using MATLAB coding [11]. We note that $\left(\frac{R}{S}\right)_t = c\tau^H$ where c is constant, τ is time span and H is Hurst exponent. R and S are defined as the following:

$R(\tau) = \max_{1\le t\le \tau} X(t,\tau) - \min_{1\le t\le \tau} X(t,\tau)$ and

$S = \left[\frac{1}{\tau}\sum_{t=1}^{\tau}\{\xi(t) - E(\xi)_\tau\}^2\right]^{\frac{1}{2}}$

where $E(\xi)_\tau = \frac{1}{\tau}\sum_{t=1}^{\tau}\xi(t)$ and $X(t,\tau) = \sum_{u=1}^{t}[\xi(u) - E(\xi)_\tau]$.

VIX (Volatility Index) is considered to be a premier barometer of investor sentiment and market volatility; is often described as the "rate and magnitude of changes in prices"; and is referred to as risk [6]. Volatility Index (calculated as annualized volatility, denoted in percentage) is a measure, of the amount by which an underlying Index is expected to fluctuate, in the near term, based on the order book of the underlying index options. We have analyzed data for the CBOE (Chicago Board Options Exchange) VIX whose underlying is S&P 500 stock index option prices; as well as the data for India VIX whose underlying is NIFTY option prices. This data is available at [9, 10].

For India VIX we have analyzed total 1000 data points from 1st June 2009 to 29th May 2013.

Table 1

Open	High	Low	Close
Hurst Exponent Values			
H_O	H_H	H_L	H_C
0.8878	0.8914	0.8873	0.8909
Fractal Dimensions			
D_O	D_H	D_L	D_C
1.1122	1.1086	1.1127	1.1091
Predictability Indices			
PI_O	PI_H	PI_L	PI_C
0.7756	0.7828	0.7746	0.7818

Predictability Index vector $PI_V = (0.7756, 0.7828, 0.7746, 0.7818)$
Norms of PI_V: $\|PI_V\|_S = 3.1148$
$\|PI_V\|_M = 0.7828$
$\|PI_V\|_E = 1.5574$

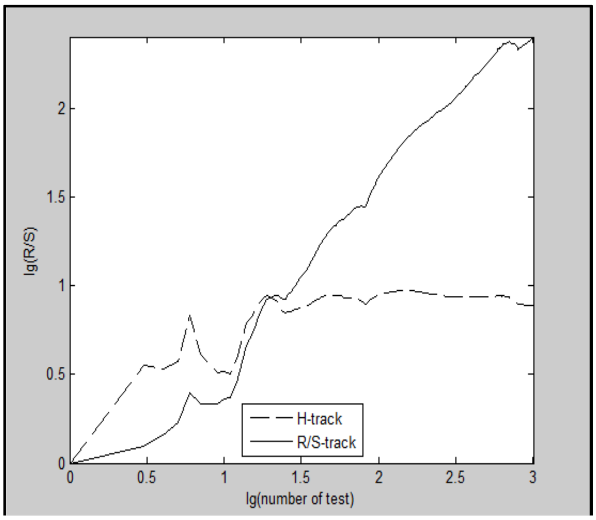

Fig 1 India VIX – Closing

For India VIX we form a time series of each data point by defining different norms and then find the Hurst value and fractal dimension.

Table 2

Sum Norm	Max Norm	Euclidean Norm
Hurst Exponent Values		
H_S	H_M	H_E
0.8899	0.8914	0.8900
Fractal Dimensions		
D_S	D_M	D_E
1.1101	1.1086	1.1100
Predictability Indices		
PI_S	PI_M	PI_E
0.7798	0.7828	0.7800

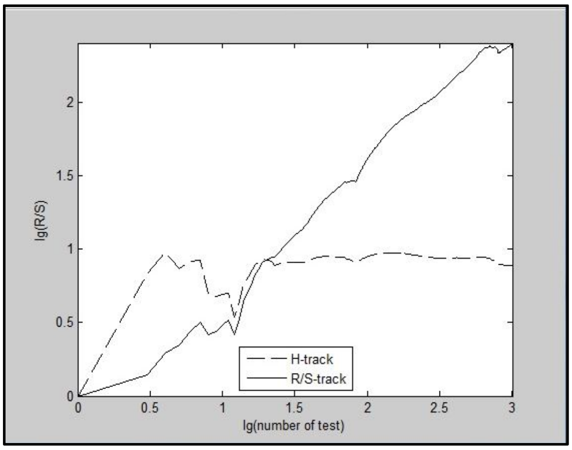

Fig 2 Euclidean Norm

For CBOE VIX we analyzed total 2365 data points from 1st February 2004 to 23rd May 2013.

Table 3

Open	High	Low	Close
Hurst Exponent Values			
H_O	H_H	H_L	H_C
0.9349	0.9335	0.9367	0.9348
Fractal Dimensions			
D_O	D_H	D_L	D_C
1.0651	1.0665	1.0633	1.0652
Predictability Indices			
PI_O	PI_H	PI_L	PI_C
0.8698	0.8670	0.8734	0.8696

Predictability Index vector $PI_V = (0.8698, 0.8670, 0.8734, 0.8696)$
Norms of PI_V: $\|PI_V\|_S = 3.4798$
$\|PI_V\|_M = 0.8698$
$\|PI_V\|_E = 1.7399$

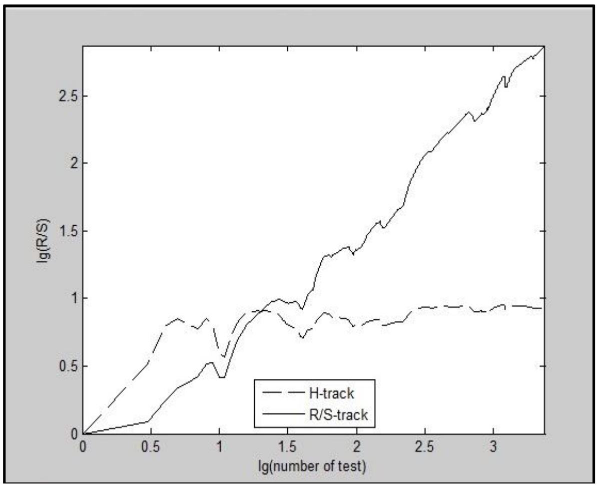

Fig 3 CBOE VIX – Closing

The calculated values of fractal dimensions reveal that each of the time series V_O, V_H, V_L, V_C follows fractional Brownian motions exhibiting persistence behavior in conformity with long range memory and resulting into non-zero predictability attribute. A comparison of predictability indices for CBOE VIX (S&P 500 Index) and India VIX (NIFTY) reveals that a more matured market like US market exhibit more predictable behavior than a complex Indian Market. This predictability analysis can be carried out for other variable parameters like stock price, option price, currency derivatives, and commodity derivatives to infer whether the underlying dynamics is BM or fBM.

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