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Fixed Point Theorems of Weakly contractive fuzzy mappings in Metric Spaces

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Abstract

We prove common fixed point theorems for a pair of Weakly contractive fuzzy mappings in a metric linear space. The presented results extend some known existence results from the literature.

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1. Introduction

Alber and Guerr-Delabriere [16] introduced the generalizations of contractions on metric spaces, is the concept of weakly contractive mappings and defined such concepts for point-to-point mappings on Hilbert spaces and proved the existence of fixed points. Rhoades [17] showed that most results of [16] are true for any Banach space. Subsequently Bae [18] obtained fixed points of multivalued weakly contractive mappings and Beg and Abbas [19] presented results regarding common fixed points and coincidence points of a pair of point-to-point mappings, one is weakly contractive with respect to other. In this paper, we obtain a fixed point theorem for a pair of fuzzy mappings satisfying a generalized weakly contractive condition.

Let (X, d) be a metric space. A mapping $T : X \to X$ is weakly contractive if $d(Tx, Ty) \leq d(x, y) - \phi(d(x, y))$ for each $x, y \in X$ where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ϕ is positive on $[0, \infty)$, $\phi(0) = 0$ and $\underset{n\to\infty}{\mathrm{Lt}}\ \phi(t) = \infty$.

Let (X, d) be a metric linear space and $F(X)$, the collection of all fuzzy sets in X. Let $A \in F(X)$ and $\alpha \in [0, 1]$. The $\alpha-$level set of A, denoted by A_α. and $A_\alpha = \{x : A(x) \geqslant \alpha\}$ and $A_0 = \overline{\{x : A(x) > 0\}}$ where $\overline{A}$ stands for the (nonfuzzy) closure of A.

Definition 1.1. A fuzzy subset A of X is an approximate quantity iff its $\sup\limits_{x\in X} A(x) = 1$.

From the collection $F(X)$, a subcollection of approximate quantities is denoted as $W(X)$. The distance between two approximate quantities is defined by the following scheme.

Let $A, B \in W(X)$, and $\alpha \in [0, 1]$,

$$D_\alpha(A, B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x, y)$$

and D_α is a nondecreasing function of α. and $H_\alpha(A, B) = \mathrm{dist}(A_\alpha, B_\alpha)$, $H(A, B) = \sup\limits_{a} D_\alpha(A, B)$, Wherein the dist is in the sense of Hausdorff distance.

The function D_α is called a $\alpha-$distance (induced by d), H_α is called a $a-$distance (induced by dist), and H a distance between A and B.

Note 1.2. Let $A, B \in W(X)$. An approximate quantity A is more accurate than B, denoted $A \subset B$ iff $A(x) \leq B(x)$, for each $x \in X$. It is clear that $\subset$ is a partial order relation determined on the family $W(X)$.

Definition 1.3. Let Y be an arbitrary set and X any metric linear space. F is called fuzzy mapping iff F is a mpping from the set Y into $W(X)$, that is $F(y) \in W(X)$ for each $y \in Y$ and the function value $F(y, x)$ stands for the grade of membership of x in $F(y)$.

Lemma 1.4. Lee and Cho [3] Let (X, d) be a complete metric linear space, $T : X \to W(X)$ be a fuzzy mapping and $x_0 \in X$. Then there exists a $x_1 \in X$ such that $x_0 \subset T(x_1)$.

Lemma 1.5. Nadler [8] Let A and B be nonempty compact subsets of a metric space (X, d). If $a \in A$, then there exists a $b \in B$ such that $d(a, b) \leq H(A, B)$.

Lemma 1.6. Hu [9] Let (X, d) be a metric space and $CB(X)$ be the family of nonempty closed and bounded subsets of X. Let $\{A_n\}$ be a sequence of sets in $CB(X)$ and $\underset{n\to\infty}{\mathrm{Lt}}\ H(A_n, A) = 0$ for $A \in CB(X)$. If $y_n \in A_n$ $(n = 1, 2, \dots)$ and $d(y_n, y) \to 0$, then $y \in A$.

Lemma 1.7. Let $x \in X$, $A \in W(x)$ and $\{x\}$ a fuzzy set with membership function equal to a characteristic function of $\{x\}$. If $\{x\} \subset A$, then $D_\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.

Lemma 1.8. $D_\alpha(x, A) \leq d(x, y) + D_\alpha(y, A)$ for each $x, y \in X$.

Lemma 1.9. If $\{x_0\} \subset A$, then $D_\alpha(x_0, B) \leq H_\alpha(A, B)$ for each $B \in W(x)$.

Lemma 1.10. If $F : X \to W(X)$ is a fuzzy mapping and $x_0 \in X$, then there exists $x_1 \in X$ such that $\{x_1\} \subset Fx_0$.

Definition 1.11. A fuzzy mapping $T : X \to W(X)$ is weakly contractive if $D(T(x), T(y)) \leq d(x, y) - \phi(d(x, y))$ for each $x, y \in X$.

2. Main results

Theorem 2.1. Let (X, d) be a complete metric space and let S, T be fuzzy mappings from X to $W(x)$ satisfy the condition

$$D(S(x), T(y)) \leq d(x, y) - \phi(\max\{d(x, y), D(x, S(x)), D(y, T(y))\})$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that ϕ is positive on $[0, \infty), \phi(0) = 0$ and $\operatorname*{Lt}_{n\to\infty} \phi(t) = \infty$. Then there exists a point $z \in X$ such that $\{z\} \subset S(z) \cap T(z)$.

Proof. Let x_0 be an arbitrary, but fixed element of X. We shall construct a sequence $\{x_n\}$ of points of X as follow: there exists $x_1 \in X$ such that $\{x_1\} \subset S(x_0)$ and $x_2 \in X$ such that $\{x_2\} \subset T(x_1)$ and

$$\begin{aligned} d(x_1, x_2) &\leq H(S(x_0), T(x_1)) \\ &\leq D(S(x_0), T(x_1)) \\ &\leq d(x_0, x_1) - \phi(\max\{d(x_0, x_1), D(x_0, S(x_0)), D(x_1, T(x_1))\}) \end{aligned}$$

Continuing this process, having chosen $x_n \in X$ we obtain $x_{n+1} \in X$ such that $\{x_{2k+1}\} \subset S(x_{2k}), \{x_{2k+2}\} \subset T(x_{2k+1})$, where

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &\leq D(S(x_{2k}), T(x_{2k+1})) \\ &\leq d(x_{2k}, x_{2k+1}) - \phi(\max\{d(x_{2k}, x_{2k+1}), D(x_{2k}, S(x_{2k})), \\ &\quad D(x_{2k+1}, T(x_{2k+1}))\}) \\ d(x_{2k+2}, x_{2k+3}) &\leq D(S(x_{2k+1}), T(x_{2k+2})) \\ &\leq d(x_{2k+1}, x_{2k+2}) - \phi(\max\{d(x_{2k+1}, x_{2k+2}), D(x_{2k+1}, S(x_{2k+1})), \\ &\quad D(x_{2k+2}, T(x_{2k+2}))\}) \end{aligned}$$

It implies that $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}), n = 0, 1, 2, \ldots$ which shows that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers and therefore tends to a limit $l \geq 0$. If $l > 0$, then we have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) - \phi(l)$.

Thus, $d(x_{n+N}, x_{n+N+1}) \leq d(x_n, x_{n+1}) - N\phi(l)$, which is a contradiction for N large enough. Therefore $d(x_n, x_{n+1}) \to 0$. By a similar argument it follows that $\{x_n\}$ is a

Cauchy sequence in X, therefore $x_n \to z \in X$. and $x_{2k+1} \to z, x_{2k+2} \to z$. Thus,

$$\begin{aligned} d(x_{2k+1}, T(z)) = H(S(x_{2k}), T(z)) &= D_1(S(x_{2k}), T(z)) \\ &\leq D(S(x_{2k}), T(z)) \\ &\leq d(x_{2k}, z) - \phi(\max\{d(x_{2k}, z), D(x_{2k}, x_{2k})), \\ &\quad D(z, T(z))\}) \end{aligned}$$

Taking the limit $k \to 0$ we get $d(z, T(z)) \leq 0$, from which it follows that $\{z\} \subset T(z)$. Similarly, $\{z\} \subset S(z)$. Hence $\{z\} \subset S(z) \cap T(z)$. ■

Corollary 2.2. Let (X, d) be a complete metric space and let S, T be fuzzy mappings from X to $W(x)$ satisfy the condition

$$D(S(x), T(y)) \leq d(x, y) - \phi(d(x, y))$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that ϕ is positive on $[0, \infty), \phi(0) = 0$ and $\underset{n\to\infty}{\mathrm{Lt}}\ \phi(t) = \infty$. Then there exists a point $z \in X$ such that $\{z\} \subset S(z) \cap T(z)$.

Theorem 2.3. Let (X, d) be a complete metric space and let T be fuzzy map from X to $W(x)$ satisfy the condition

$$D(T(x), T(y)) \leq d(x, y) - \phi(\max\{d(x, y), D(x, T(x)), D(y, T(y))\})$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that ϕ is positive on $[0, \infty), \phi(0) = 0$ and $\underset{n\to\infty}{\mathrm{Lt}}\ \phi(t) = \infty$. Then there exists a point $z \in X$ such that $\{z\} \subset T(z)$.

Proof. Let x_0 be an arbitrary, but fixed element of X. We shall construct a sequence $\{x_n\}$ of points of X as follow: For every $x_0 \in X$ and $n \geq 1$, $\{x_1\} \subset T(x_0)$.

$$\begin{aligned} d(x_1, x_2) &\leq H(T(x_0), T(x_1)) \\ &\leq D(T(x_0), T(x_1)) \\ &\leq d(x_0, x_1) - \phi(\max\{d(x_0, x_1), D(x_0, T(x_0)), D(x_1, T(x_1))\}) \end{aligned}$$

Continuing this process, having chosen $x_n \in X$ we obtain $x_{n+1} \in X$ such that $\{x_{n+1}\} \subset T(x_n)$ where

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq D(T(x_n), T(x_{n+1})) \\ &\leq d(x_n, x_{n+1}) - \phi(\max\{d(x_n, x_{n+1}), D(x_n, T(x_n)), D(x_{n+1}, T(x_{n+1}))\}) \end{aligned}$$

It implies that $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}), n = 0, 1, 2, \ldots$ which shows that $\{d(x_n, x_{n+1})\}$ is a non-increasing sequence of positive real numbers and therefore tends to a limit $l \geq 0$. If $l > 0$, then we have $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1}) - \phi(l)$.

Thus, $d(x_{n+N}, x_{n+N+1}) \leq d(x_n, x_{n+1}) - N\phi(l)$, which is a contradiction for N large enough. Therefore $d(x_n, x_{n+1}) \to 0$. By a similar argument it follows that $\{x_n\}$ is a Cauchy sequence in X, therefore $x_n \to z \in X$. And $x_{k+1} \to z$. Thus,

$$\begin{aligned} d(x_{k+1}, T(z)) &= H(T(x_k), T(z)) \\ &\leq D(T(x_k), T(z)) \\ &\leq d(x_k, z) - \phi(\max\{d(x_k, z), D(x_k, x_k)), D(z, T(z))\}) \end{aligned}$$

Taking the limit $k \to 0$ we get $d(z, T(z)) \leq 0$, from which it follows that $z \subset T(z)$. ■

Corollary 2.4. Let (X, d) be a complete metric space and let T be fuzzy map from X to $W(x)$ satisfy the condition

$$D(T(x), T(y)) \leq d(x, y) - \phi(d(x, y))$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that ϕ is positive on $[0, \infty), \phi(0) = 0$ and $\underset{n\to\infty}{\mathrm{Lt}} \phi(t) = \infty$. Then there exists a point $z \in X$ such that $\{z\} \subset T(z)$.

Theorem 2.5. Let (X, d) be a complete metric space and let T be fuzzy map from X to $W(x)$ satisfy the condition $D(T(x), T(y)) \leq kd(x, y)$ for all $x, y \in X$, where $k \in (0, 1)$. Then there exists a point $z \in X$ such that $\{z\} \subset T(z)$.

Proof. Let x_0 be an arbitrary element of X. and $\{x_1\} \subset T(x_0)$. then there exists $\{x_2\} \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq H(T(x_0), T(x_1)) \\ &\leq D(T(x_0), T(x_1)) \\ &\leq kd(x_0, x_1) + k \end{aligned}$$

similarly, there is $\{x_3\} \in Tx_2$ such that

$$\begin{aligned} d(x_3, x_2) &\leq H(Tx_2, Tx_1) \leq D(Tx_2, Tx_1) \\ &\leq kd(x_2, x_1) + k \\ &\leq k^2 d(x_0, x_1) + 2k^2 \end{aligned}$$

By induction, we get a sequence $\{x_n\}$ such that $\{x_{n+1}\} \subset Tx_n$ and

$$d(x_{n+1}, x_n) \leq k^n d(x_1, x_0) + nk^n.$$

Thus, $\{x_n\}$ is a cauchy sequence. Since X is complete, let $x_n \to z$. Now

$$\begin{aligned} d(z, Tz) &\leq d(z, x_n) + d(x_n, Tz) \\ &\leq d(p, x_n) + H(Tx_{n-1}, Tz) \\ &\leq d(p, x_n) + kd(x_{n-1}, z) \to 0. \end{aligned}$$

Hence $\{z\} \subset T(z)$. ■

Theorem 2.6. Let (X, d) be a complete metric space and let S, T be fuzzy map from X to $W(x)$ satisfy the condition

$$D(Sx, Ty) \leq \frac{k}{\sqrt{2}}\{D(x, Sx)D(y, Ty) + D(x, Ty)D(Sx, y) + D(y, Ty)d(x, y)\}^{\frac{1}{2}}$$

for all $x, y \in X$, where $k \in (0, 1)$. Then S and T have a common fixed point.

Proof. Let x_0 be an arbitrary element of X. and $\{x_1\} \subset S(x_0)$ then there exists $\{x_2\} \in Tx_1$ such that

$$d(x_1, x_2) \leq \frac{1}{\sqrt{k}}D_1(Sx_0, Tx_1) \leq \frac{1}{\sqrt{k}}D(Sx_0, Tx_1)$$

$$\begin{aligned}
d(x_1, x_2) &\leq \frac{1}{\sqrt{k}}D_1(Sx_0, Tx_1) \\
&\leq \frac{1}{\sqrt{k}}D(Sx_0, Tx_1) \\
&\leq \frac{\sqrt{k}}{\sqrt{2}}\{D(x_0, Sx_0)D(x_1, Tx_1) + D(x_0, Tx_1)D(Sx_0, x_1) + D(x_1, Tx_1)d(x_0, x_1)\}^{\frac{1}{2}} \\
&\leq \frac{\sqrt{k}}{\sqrt{2}}\{d(x_0, x_1)d(x_1, x_2) + d(x_0, x_2)d(x_1, x_1) + d(x_1, x_2)d(x_0, x_1)\}^{\frac{1}{2}} \\
&\leq \sqrt{k}\{d(x_0, x_1)d(x_1, x_2)\}^{\frac{1}{2}} \\
d(x_1, x_2) &\leq kd(x_0, x_1).
\end{aligned}$$

Similarly, we can choose $x_3 \in X$ such that $\{x_3\} \subset Sx_2$ and $d(x_2, x_3) \leq kd(x_1, x_2)$. In general, we have that for $n = 0, 1, 2, \cdots$

$$d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1})$$

and $\{x_{2n-1}\} \subset Sx_{2n-2}, \{x_{2n}\} \subset Tx_{2n-1}$ with

$$d(x_{2n-1}, x_{2n}) \leq \frac{1}{\sqrt{k}}D_1(Sx_{2n-2}, Tx_{2n-1}) \leq \frac{1}{\sqrt{k}}D(Sx_{2n-2}, Tx_{2n-1})$$

and so $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$ and hence $\{x_n\}$ is Cauchy sequence in X. By completeness of X, the sequence $\{x_n\}$ converges to some $z \in X$. From lemmas, for each

$\alpha \in [0, 1]$, we have that

$$
\begin{aligned}
D_\alpha(z, Tz) &\le d(z, x_{2n}) + D_\alpha(x_{2n}, Sz) \\
&\le d(z, x_{2n}) + D_1(Sz, Tx_{2n-1}) \\
&\le d(z, x_{2n}) + D(Sz, Tx_{2n-1}) \\
&\le d(z, x_{2n}) + \frac{k}{\sqrt{2}}\{D(z, Sz)D(x_{2n-1}, Tx_{2n-1}) + \\
&D(z, Tx_{2n-1})D(Sz, x_{2n-1}) + D(x_{2n-1}, Tx_{2n-1})d(z, x_{2n-1})\}^{\frac{1}{2}} \\
&\le d(z, x_{2n}) + \frac{k}{\sqrt{2}}\{D(z, Sz)d(x_{2n-1}, x_{2n}) + \\
&d(z, x_{2n})d(Sz, x_{2n-1}) + d(x_{2n-1}, x_{2n})d(z, x_{2n-1})\}^{\frac{1}{2}}
\end{aligned}
$$

Taking limit $n \to \infty$, $D_\alpha(z, Sz) = 0$ and hence $\{z\} \subset Sz$. Similarly, we can show $\{z\} \subset Tz$. ■

Corollary 2.7. Let (X, d) be a complete metric space and let S, T be fuzzy map from X to $W(x)$ satisfy the condition

$$D(Sx, Ty) \le \frac{k}{\sqrt{2}}\{D(x, Sx)D(y, Ty) + D(y, Ty)d(x, y)\}^{\frac{1}{2}}$$

for all $x, y \in X$, where $k \in (0, 1)$. Then S and T have a common fixed point.

Corollary 2.8. Let $S, T : X \to W(X)$ be fuzzy mappings satisfying the following condition: There exists $k \in (0, 1)$ such that $D(Sx, Ty) \le k\{p(x, Sx)p(y, Ty)p(y, Ty)d(x, y)\}^{\frac{1}{4}}$ for all $x, y \in X$. Then S and T have a common fixed point.

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