Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 5, Number 4 (2013), pp. 229-236

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Pell and Pell-Lucas Identities

Naresh Patel* and Punit Shrivastava**

**Assistant Professor (Mathematics)*
Government College Jobat
Distt. – Alirajpur (M.P.) India.
E-mail: n_patel_1978@yahoo.co.in
***Lecturer (Mathematics)*
Dhar Polytechnic College, Dhar (M.P.) India
E-mail: puneetsri2001@yahoo.co.in

ABSTRACT:

In this paper, we have discussed some Pell and Pell-Lucas identities with proofs of some identities using their Binet form. These properties can be used to derive generating functions, polynomials, divisibility properties, matrices, determinants and so many other applications of Pell and Pell-Lucas Sequences.

MSC 2000 Classification: 11B39

Key words: Identities, Binet Forms, Pell and Pell-Lucas sequences.

1. INTRODUCTION:

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers *n* by

$$(1.1) \qquad \begin{cases} U_n = pU_{n-1} + U_{n-2}, & U_0 = 0,\ U_1 = 1, \\ V_n = pV_{n-1} + V_{n-2}, & V_0 = 2,\ V_1 = p. \end{cases}$$

For p = 1, we write $\{U_n\} = \{F_n\}$ and $\{V_n\} = \{L_n\}$, which are the Fibonacci and Lucas numbers respectively. Their Binet forms, obtained by using standard techniques for solving linear recurrences, are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where α and β are the roots of $x^2 - x - 1 = 0$.

For p = 2, we write

(1.2) $$\begin{cases} P_n = 2P_{n-1} + P_{n-2}, & P_0 = 0,\ P_1 = 1, \\ Q_n = 2Q_{n-1} + Q_{n-2}, & Q_0 = 2,\ Q_1 = 2. \end{cases}$$

Here $\{P_n\}$ and $\{Q_n\}$ are the Pell and Pell-Lucas Sequences respectively. Their Binet forms are given by

(1.3) $$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ and } Q_n = \gamma^n + \delta^n,$$

where γ and δ are the roots of $x^2 - 2x - 1 = 0$ i.e. $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$,

(1.4) $$\therefore \quad \gamma\delta = -1, \gamma = \frac{-1}{\delta}, \delta = \frac{-1}{\gamma}.$$

2. IDENTITIES:

The Pell and Pell-Lucas identities are given below. For brevity we have given proof to some identities only and others can be proved in similar ways using Binet forms i.e. (1.3).

$$Q_n = P_{n+1} + P_{n-1} \tag{2.1}$$

Proof: By (1.3), we can write

$$P_{n+1} + P_{n-1} = \frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} + \frac{\gamma^{n-1} - \delta^{n-1}}{\gamma - \delta} = \frac{\gamma^{n+1} - \delta^{n+1} + \gamma^n\left(\frac{1}{\gamma}\right) - \delta^n\left(\frac{1}{\delta}\right)}{\gamma - \delta}$$

$$= \frac{\gamma^{n+1} - \delta^{n+1} + \gamma^n(-\delta) - \delta^n(-\gamma)}{\gamma - \delta} \qquad \text{By (1.4)}$$

$$= \frac{\gamma^n(\gamma - \delta) + \delta^n(\gamma - \delta)}{\gamma - \delta} = \frac{(\gamma - \delta)(\gamma^n + \delta^n)}{\gamma - \delta}$$

$$= \gamma^n + \delta^n = Q_n \qquad \text{By (1.3)}$$

$$Q_n = 2(P_n + P_{n-1}) \tag{2.2}$$

Proof: By (2.1) and (1.2), result is obvious.

$$2Q_n = P_{n+2} - P_{n-2}$$

Proof: Using (2.2) and (1.2), result can easily be proved.

$$Q_{n-1} + Q_{n+1} = 8P_n$$

Proof: Using (2.1) and (1.2), result can easily be proved.

$$\gamma^n = \gamma P_n + P_{n-1} \tag{2.3}$$

Proof: This result can be proved using Principle of Mathematical Induction.
Let the result be true for positive integers less than or equal to *k* i.e.

$$\gamma^k = \gamma P_k + P_{k-1} \tag{2.4}$$

Now $\gamma^{k+1} = \gamma.\gamma^k$

$= \gamma(\gamma P_k + P_{k-1})$ By (2.4)

$= \gamma^2 P_k + \gamma P_{k-1}$

$= (2\gamma + 1)P_k + \gamma P_{k-1}$ By (2.4)

$= \gamma(2P_k + P_{k-1}) + P_k$

$= \gamma P_{k+1} + P_k$ By (1.2)

So the result is true for *k+1*also, hence by strong version of Principle of Mathematical Induction result is proved.

$$\delta^n = \delta Q_n + Q_{n-1} \tag{2.5}$$

Proof: Similar to the proof of identity (v), one can prove it.

$$P_{2n} = P_n Q_n \tag{2.6}$$

Proof: By (1.3), we can write

$$P_n Q_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}\left(\gamma^n + \delta^n\right) = \frac{\gamma^{2n} - \delta^{2n}}{\gamma - \delta} = P_{2n}$$

$$P_{2^m} = Q_{2^{m-1}} Q_{2^{m-2}} \ldots\ldots Q_4 Q_2 Q_1$$

Proof: Using (2.6), we can write

$$P_{2^m} = P_{2(2^{m-1})} = P_{2^{m-1}} Q_{2^{m-1}} = Q_{2^{m-1}}(P_{2^{m-2}} Q_{2^{m-2}}) = Q_{2^{m-1}} Q_{2^{m-2}}(P_{2^{m-3}} Q_{2^{m-3}})$$

Continuing like this, we get

$$P_{2^m} = Q_{2^{m-1}} Q_{2^{m-2}} \ldots\ldots Q_4 Q_2 Q_1.$$

$$P_{-n} = (-1)^{n+1} P_n \tag{2.7}$$

Proof: By (1.3), we can write

$$P_{-n} = \frac{\gamma^{-n} - \delta^{-n}}{\gamma - \delta} = \frac{\frac{1}{\gamma^n} - \frac{1}{\delta^n}}{\gamma - \delta} = \frac{\frac{\delta^n - \gamma^n}{(\gamma\delta)^n}}{\gamma - \delta}$$

$= \dfrac{(-\delta)^n - (-\gamma)^n}{\gamma - \delta}$ By (1.4)

$$=(-1)^{n+1}\frac{\gamma^n-\delta^n}{\gamma-\delta}=(-1)^{n+1}P_n \qquad \text{By (1.3)}$$

$$Q_{-n}=(-1)^n Q_n$$

Proof: Similar to the proof of identity (ix), one can prove it.

$$\gamma^{-n}=\begin{cases}\gamma P_n-P_{n+1}, & if\ n\ is\ odd\\ P_{n+1}-\gamma P_n, & Otherwise.\end{cases}$$

Proof: Since $\gamma^{-n}=\gamma P_{-n}+P_{-n-1}$ By (2.3)

$$=\gamma(-1)^{n+1}P_n+(-1)^{n+2}P_{n+1} \qquad \text{By (2.7)}$$

$$=(-1)^{n+1}(\gamma P_n-P_{n+1})=\begin{cases}\gamma P_n-P_{n+1}, & if\ n\, is\, odd\\ P_{n+1}-\gamma P_n, & Otherwise.\end{cases}$$

$$\delta^{-n}=\begin{cases}\delta P_n-P_{n+1}, & if\ n\, is\, odd\\ P_{n+1}-\delta P_n, & Otherwise.\end{cases}$$

Proof: Similar to the proof of identity (xi), one can prove it.

$$\sum_{i=0}^{n}P_{ki+j}=\begin{cases}\dfrac{P_{nk+k+j}-(-1)^k P_{nk+j}-P_j-(-1)^j P_{k-j}}{Q_k-(-1)^k-1}, & if\ j<k\\[2ex] \dfrac{P_{nk+k+j}-(-1)^k P_{nk+j}-P_j+(-1)^k P_{j-k}}{Q_k-(-1)^k-1}, & Otherwise.\end{cases}$$

Proof: $\sum_{i=0}^{n}P_{ki+j}=\sum_{i=0}^{n}\frac{\gamma^{ki+j}-\delta^{ki+j}}{\gamma-\delta}$ By(1.3)

$$=\frac{1}{\gamma-\delta}\left\{\gamma^j\sum_{i=0}^{n}\gamma^{ki}-\delta^j\sum_{i=0}^{n}\delta^{ki}\right\}$$

$$=\frac{1}{\gamma-\delta}\left\{\gamma^j\left(\frac{\gamma^{nk+k}-1}{\gamma^k-1}\right)-\delta^j\left(\frac{\delta^{nk+k}-1}{\delta^k-1}\right)\right\} \qquad \text{(Using sum of a G.P.)}$$

$$=\frac{1}{\gamma-\delta}\left\{\frac{\gamma^j(\gamma^{nk+k}-1)(\delta^k-1)-\delta^j(\delta^{nk+k}-1)(\gamma^k-1)}{(\gamma^k-1)(\delta^k-1)}\right\}$$

$$=\frac{1}{\gamma-\delta}\left\{\frac{-(\gamma^{nk+k+j}-\delta^{nk+k+j})+(\gamma^k\delta^j-\gamma^j\delta^k)+(\gamma^j-\delta^j)+(\gamma\delta)^k(\gamma^{nk+j}-\delta^{nk+j})}{(\gamma\delta)^k-(\gamma^k+\delta^k)+1}\right\}$$

$$=\frac{-P_{nk+k+j}+\left(\dfrac{\gamma^k\delta^j-\gamma^j\delta^k}{\gamma-\delta}\right)+P_j+(-1)^k P_{nk+j}}{(-1)^k-Q_k+1} \qquad \text{By (1.3) and (1.4)}$$

But,

$$\gamma^k\delta^j-\gamma^j\delta^k=\begin{cases}(\gamma\delta)^j\left(\gamma^{k-j}-\delta^{k-j}\right), & if\ \ j<k\\ (\gamma\delta)^k\left(\delta^{j-k}-\gamma^{j-k}\right), & Otherwise.\end{cases}$$

Or $\dfrac{\gamma^k\delta^j-\gamma^j\delta^k}{\gamma-\delta}=\begin{cases}(-1)^j P_{k-j}, & if\ \ j<k\\ (-1)^{k+1} P_{j-k}, & Otherwise.\end{cases}$ By (1.3) and (1.4)

$$\therefore\ \sum_{i=0}^{n} P_{ki+j}=\begin{cases}\dfrac{P_{nk+k+j}-(-1)^k P_{nk+j}-P_j-(-1)^j P_{k-j}}{Q_k-(-1)^k-1}, & if\ \ j<k\\ \dfrac{P_{nk+k+j}-(-1)^k P_{nk+j}-P_j+(-1)^k P_{j-k}}{Q_k-(-1)^k-1}, & Otherwise.\end{cases}$$

$$\sum_{i=0}^{n} P_{ki}=\frac{P_{nk+k}-(-1)^k P_{nk}-P_k}{Q_k-(-1)^k-1}$$

Proof: Letting j = 0, in identity (xiii), we get the result.

$$\sum_{i=0}^{n} P_{i+j}=\begin{cases}P_{n+1+j}+P_{n+j}-P_j-(-1)^j P_{1-j}, & if\ \ j<1\\ P_{n+1+j}+P_{n+j}-P_j-P_{j-1}, & Otherwise.\end{cases}$$

Proof: Letting k = 1, in identity (xiii), we get the result.

$$\gamma^m P_{n-m+1}+\gamma^{m-1}P_{n-m}=\gamma^n$$

Proof: Since $\gamma^m P_{n-m+1}+\gamma^{m-1}P_{n-m}=\gamma^{m-1}(\gamma P_{n-m+1}+P_{n-m})=\gamma^{m-1}(\gamma^{n-m+1})$ By (2.3)

$=\gamma^n$

$$\delta^m P_{n-m+1}+\delta^{m-1}P_{n-m}=\delta^n$$

Proof: Similar to the proof of identity (xvi) using (2.5), one can prove it.

$$P_m Q_n+P_n Q_m=2P_{m+n}$$

Proof: Since $P_m Q_n+P_n Q_m=\left(\dfrac{\gamma^m-\delta^m}{\gamma-\delta}\right)\left(\gamma^n+\delta^n\right)+\left(\dfrac{\gamma^n-\delta^n}{\gamma-\delta}\right)\left(\gamma^m+\delta^m\right)$ By (1.3)

$$=\frac{(\gamma^{m+n}+\gamma^m\delta^n-\delta^m\gamma^n-\delta^{m+n})+(\gamma^{m+n}+\gamma^n\delta^m-\delta^n\gamma^m-\delta^{m+n})}{\gamma-\delta}$$

$=\dfrac{2(\gamma^{m+n}-\delta^{m+n})}{\gamma-\delta}=2P_{m+n}$ By (1.3)

Similar to the proof of identity (xviii) using (1.3), we can prove the following identities:

$$Q_n=Q_m P_{n-m+1}+Q_{m-1}P_{n-m}$$

$$P_k Q_{n+j} - P_j Q_{n+k} = (-1)^j P_{k-j} Q_n$$
$$P_{mn} = Q_m P_{m(n-1)} + (-1)^{m+1} P_{m(n-2)}$$
$$P_{3n} = Q_n P_{2n} - (-1)^n P_n$$
$$P_{3n} = P_n \{Q_{2n} + (-1)^n\}$$
$$Q_{3n} = Q_n \{Q_{2n} - (-1)^n\}$$
$$P_{m+n} = P_m Q_n - (-1)^n P_{m-n}$$
$$Q_m Q_n + 8 P_n P_m = 2 Q_{m+n}$$
$$P_{n+1} Q_{n+2} - P_{n+2} Q_n = P_{2n+1} + P_{2n+2} - 1$$
$$P_{4n+3} - 1 = Q_{2n+1} P_{2n+2}$$
$$P_{4n+1} - 1 = Q_{2n+1} P_{2n}$$
$$Q_n Q_{n+1} = Q_{2n+1} + 2(-1)^n$$
$$Q_{n+2} - Q_{n-2} = 16 P_n$$
$$P_{2m+n} - (-1)^m P_n = P_m Q_{m+n}$$
$$P_{2m+n} + (-1)^m P_n = P_{m+n} Q_m$$
$$Q_{(2m+1)(4n+1)} - Q_{2m+1} = 8 P_{2n(2m+1)} P_{(2m+1)(2n+1)}$$
$$Q_{m+r} Q_{m+r+1} + Q_{m-r} Q_{m-r+1} = Q_{2m+1} Q_{2r} + 4(-1)^{m+r}$$
$$Q_{m+r} Q_{m+r+1} + Q_{m-r} Q_{m-r+1} = Q_{2m+2r+1} + Q_{2m-2r+1} + 4(-1)^{m+r}$$
$$8(P_{m+r} P_{m+r+1} + P_{m-r} p_{m-r+1}) = Q_{2m+1} Q_{2r} - 4(-1)^{m+r}$$
$$P_n Q_{n+r} - Q_n Q_{n-r} = P_{2n+r} - Q_{2n-r} - (-1)^n \{P_r + (-1)^r Q_r\}$$
$$P_n Q_{n+r} - P_{n+r} Q_{n-r} = (P_{2r} - P_r) P_{2n-r+1} + (P_{2r-1} - P_{r-1}) P_{2n-r} - (-1)^n \{P_r + (-1)^r P_{2r}\}$$

Now, identities involving squares of P_n and Q_n are given below:

$$8 P_n^2 = Q_n^2 - 4(-1)^n$$

Proof: Using (1.3) and (1.4), we can write

$$Q_n^2 - 4(-1)^n = (\gamma^n + \delta^n)^2 - 4(\gamma\delta)^n = (\gamma^{2n} + \delta^{2n} + 2\gamma^n\delta^n) - 4(\gamma^n\delta^n)$$
$$= (\gamma^{2n} + \delta^{2n} - 2\gamma^n\delta^n) = (\gamma^n - \delta^n)^2 = \left(\frac{\gamma^n - \delta^n}{\gamma - \delta}\right)^2 (\gamma - \delta)^2$$
$$= 8 P_n^2 \qquad (\text{Since } \gamma - \delta = 2\sqrt{2})$$

Similar to the proof of identity (xlii) using (1.3) and (1.4), we can prove the following identities:

$$Q_{2m} Q_{2n} = Q_{m+n}^2 + 8 P_{m-n}^2$$
$$Q_{2m} Q_{2n} = 8 P_{m+n}^2 + Q_{m-n}^2$$
$$Q_{2m} Q_{2n} = Q_{m+n}^2 + Q_{m-n}^2 - 4(-1)^{m+n}$$

$$Q_{4n} = 8P_{2n}^2 + 2$$
$$Q_{4n+2} = 8P_{2n+1}^2 - 2$$
$$Q_n^2 + Q_{n+1}^2 = Q_{2n} + Q_{2n+2}$$
$$P_{m+n}^2 Q_{m+n}^2 - P_m^2 Q_m^2 = P_{2n} P_{4m+2n}$$
$$\left(Q_{n+r}^2 + Q_{n-r}^2\right) = Q_{2n} Q_{2r} + 4(-1)^{n+r}$$
$$8\left(P_{n+r}^2 + P_{n-r}^2\right) = Q_{2n} Q_{2r} - 4(-1)^{n+r}$$
$$Q_n^2 + Q_{n+1}^2 = 8P_{2n+1}$$
$$Q_{n-1} Q_{n+1} - Q_n^2 = 8(-1)^{n-1}$$
$$Q_n^2 = Q_{2n} + 2(-1)^n$$
$$Q_{2n} = 8P_n^2 + 2(-1)^n$$

Other identities, which can also be proved using Binet form for Pell and Pell-Lucas numbers are given below:

$$P_{n+k}^3 - Q_{3k} P_n^3 + (-1)^k P_{n-k}^3 = 3(-1)^n P_n P_k P_{2k}$$
$$P_{n+k}^3 + (-1)^k P_{n-k}\left(P_{n-k}^2 + 3P_{n+k} P_n Q_k\right) = Q_k^3 P_n^3$$
$$Q_{3^n} = 2\prod_{k=0}^{n-1}\left(Q_{2.3^k} + 1\right)$$
$$P_{3^n} = \prod_{k=0}^{n-1}\left(Q_{2.3^k} - 1\right)$$
$$P_m P_{2m} P_{3r} = P_{m+r}^3 - (-1)^r P_{m-r}^3 (-1)^m Q_m P_r^3$$
$$P_{2n-2} < P_n^2 < P_{2n-1}$$
$$P_{2n-1} < Q_{n-1}^2 < P_{2n}, \quad n > 2.$$
$$Q_{m+n} + Q_{m-n} = \begin{cases} 8P_m P_n, & if\ n\ is\ odd \\ Q_m Q_n, & Otherwise. \end{cases}$$
$$Q_{m+n} - Q_{m-n} = \begin{cases} Q_m Q_n, & if\ n\ is\ odd \\ 8P_m P_n, & Otherwise. \end{cases}$$
$$P_{m+n} + P_{m-n} = \begin{cases} Q_m P_n, & if\ n\ is\ odd \\ Q_n P_m, & Otherwise. \end{cases}$$
$$P_{m+n} - P_{m-n} = \begin{cases} P_m Q_n, & if\ n\ is\ odd \\ Q_m P_n, & Otherwise. \end{cases}$$
$$\frac{Q_{m+n} + Q_{m-n}}{P_{m+n} + P_{m-n}} = \begin{cases} \dfrac{8P_m}{Q_m}, & if\ n\ is\ odd \\ \dfrac{Q_m}{P_m}, & Otherwise. \end{cases}$$

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