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Packing by Edge Disjoint Trees

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Abstract

A Graph G is said to be a packing by edge disjoint trees, if its edges can be partitioned into trees. In this paper, we discuss the packability of graphs of various types by edge disjoint tree and find its number w (edt G) and its least number w_l (edt G)

Key Words: *Packing*, *Edge disjoint trees.*

1. Introduction:

By a graph $G = (V, E)$ we mean a finite, undirected, connected graph with no loop or multiple edges. The order and size of G are denoted by 'm' and 'n' respectively. For basic graph theoretic terminology, we refer to **[1, 2, 3].** The concept of random packability was introduced by Sergio Ruiz **[4]** under the name of 'random decomposable graphs'. Ruiz **[4]** obtained a characterization of randomly F- packable graphs, when F is P_3 or K_2. Lowell W Beineke **[5]** and Peter Hamburger **[5]** and Wayne D Goddard **[5]** characterized F-packable graphs where F is K_n, P_4, P_5, or P_6. S.Arumugham **[6]** and S.Meena **[6]** extended a characterization of the random packability of two or more disconnected graphs like $K_n \cup K_{1, n}$, $C_4 \cup P_2$,$3K_2$ .Elachini V. Lal **[7]** and Karakkattu S. Parvathy **[7]** characterized the class of graphs that are randomly F – packable where $F = C_4 \cup K_{1, 2} / C_5 \cup P_5 / C_4 \cup C_3 / C_4 \cup P_{2n+1}$ $(n \geq 2)$ / $C_r \cup P_{2n+1}$ $(r \geq 5,\ n \geq 2)$ / $C_r \cup P_{2n}$ $(r \geq 4, n \geq 2)$.

2. Packing by edge disjoint tree.

Definition 1: A graph G is said to be a packing by edge disjoint trees if its edges can be partitioned into trees and we denote the number of edge disjoint tree by w (edt G). Its least number is denoted by w_l (edt G).

The order and size of edge disjoint tree T are denoted by v (*edt* T) and e (*edt* T) respectively.

Example 1: Consider the following graphs; Cycle graph of order three **(**G_1), Tetrahedron (G_2) and Octahedron (G_3) are shown in figure 1.

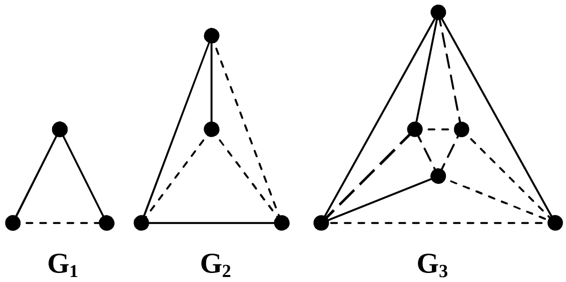

Figure 1: *Example of 2, 3, 4 - regular graphs and their packing*

In figure 1, $w_l(edt\ G_1) = 2$, $w_l(edt\ G_2) = 2$ and $w_l(edt\ G_3) = 3$

Example 2: Consider the following graphs; Herschel graph **(**G_4), G_5, G_6 and G_7 shown in figure 2.

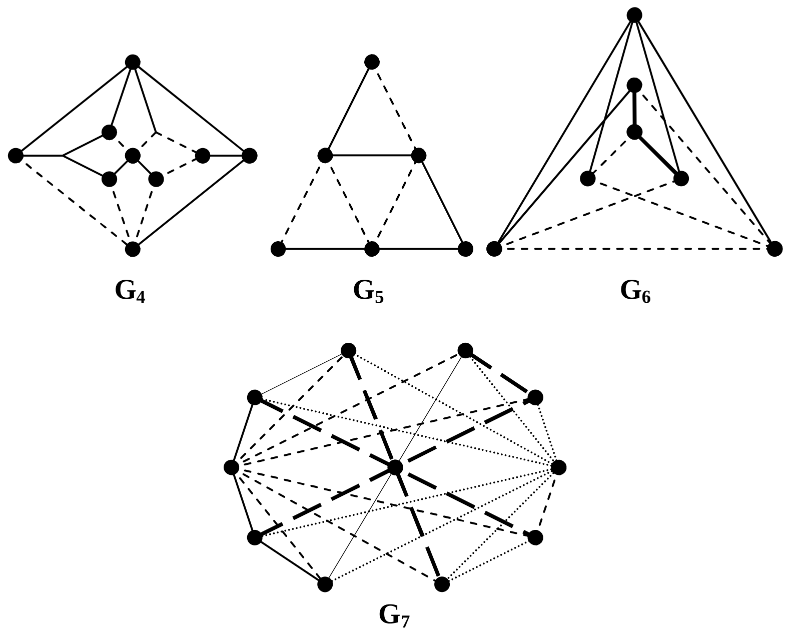

Figure 2:

$\delta\ (G_4) = 3$, *and* , $w_l\ (edt\ G_4) = 2$. $\delta\ (G_5) = 2$, *and* , $w_l\ (edt\ G_5) = 2$,
$\delta\ (G_6) = 3$, *and* , $w_l\ (edt\ G_6) = 3$ and $\delta(G_7) = 4$, *and* , $w_l\ (edt\ G_7) = 4$

Basic Packability Lemma: *Let G be a graph of size m. If G is packable by edge disjoint trees $T_1, T_2, T_3 \ldots, T_n$, ie $w\ (edt\ G) = n$, with $e(edt\ T_i) = k_i$. Then $\Sigma\, k_i = m$.*

Theorem 2.1: *Let G be* any cycle. *Then $w_l\ (edt\ G) = 2$.*

Proof: Since G is a cycle, G can't be packable by a single tree. Hence $w\ (edt\ G) > 1$. Let us assume that G is isomorphic to C_n with $v_1, v_2, \ldots, v_n\ v_1$ as vertices. Then this path must be decomposed into two or more mutually edge disjoint trees. Hence $w_l\ (edt\ G) = 2$.

Theorem 2.2: *Let G be a connected graph consisting of t blocks each isomorphic to c_n having exactly one cut vertex ($n \geq 3$, $t \geq 1$). Then $w_l\ (edt\ G) = 2$.*

Proof: By Theorem 2.1, $w\ (edt\ G) > 1$. In G, each block is connected through a single vertex. There fore, it should be packable by at least two edge disjoint trees. Hence, $w_l\ (edt\ G) = 2$.

Theorem 2.3: *A connected graph G (not regular) with $\delta(G) = 2$ having at least one cycle, then $w_l\ (edt\ G) = 2$ and , $\delta(G) = n$, ($n \geq 3$), then $w_l\ (edt\ G) = n$.*

Proof: If $\delta\ (G) = 2$, with at east one cycle, then by Theorem 2.1 $w_l\ (edt\ G) = 2$.Since, $\delta\ (G) = n$ ($n \geq 3$), then there exists at lest one vertex $v \in G$ such that $d\ (v) = n$ and all other vertices are of degree greater than n. Then, $w\ (edt\ G) \geq$ n. For, if it were packable by a number less than n, it would form a cycle. There fore, $w_l\ (edt\ G) = n$.

Theorem 2.4: *The complete bipartite graph $K_{m,n}$ is packable by at least p edge disjoint trees, where $p = min\ \{m, n\}$, ie , $w_l\ (edt\ K_{m,n}) = p$.*

Proof: The graph $K_{m,n}$ *has* order $m + n$ and size mn. It is packable by p star graphs $K_{1,r}$ *of* size r, where $r = max\ \{m, n\}$ and $p = min\ \{m, n\}$. For, if it were packable by a number less than p, it would form a cycle. This completes the proof.

Theorem 2.5: *For the complete graph K_n*

$$w_l\ (edt\ K_n) = \begin{cases} 1 & , \text{ if } n = 1, 2 \\ 2 & , \text{ if } n = 3 \\ n - 2 & , \text{ if } n \geq 4 \end{cases}$$

Proof: For the graph of order 1 and 2, it is clear that the graph is packed with a unique tree. If the graph is of order three, it follows from Theorem 2.1.

The complete graph $K_{n,}$ ($n \geq 4$) has size n (n – 1) / 2. In K_n, greatest tree is of size ($n - 1$), i.e., P_n. For, otherwise, it would form a cycle. The next largest tree is the star graph $K_{1, n-2}$. Proceeding like this, we get trees of the form $K_{1, n-3}$, $K_{1, n-4}$, ...,$K_{1, 4}$, $K_{1, 3}$ $K_{1, 2}$, $K_{1,1}$. But we cannot combine any two of these except $K_{1, 2}$ and $K_{1, 1}$. Hence w_1 ($edt\ K_n$) = n – 2 with e ($edt\ T_i$) are $n - 1$, $n - 2$,... , 4, 3, 3. Clearly, combining these $n - 2$ trees form the graph of size n $(n - 1) / 2$. Hence the proof.

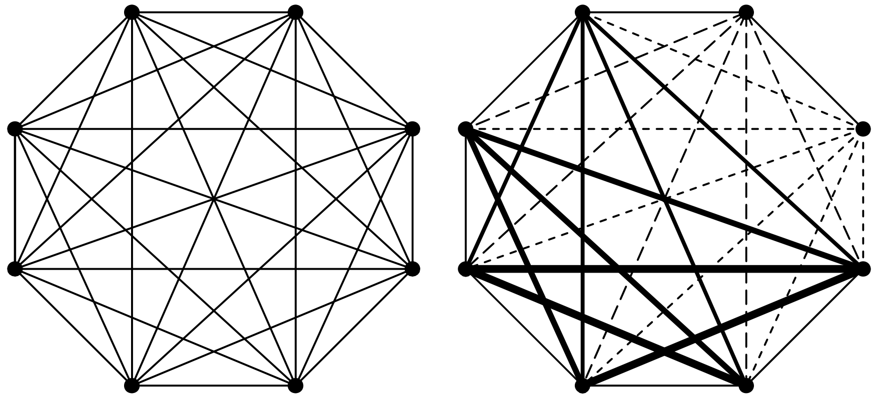

Figure 3: *K_n (n = 8), 8 – 2 = 6 edge disjoint trees of size 7, 6, 5, 4, 3, 3*

Theorem 2.6: *For the complete bipartite graph $K_{n, n}$*

$$w_l\,(edt\,K_{n,\,n}) = \begin{cases} n, & \text{if } n = 1, 2 \\ n - 1, & \text{if } n \geq 3 \end{cases}$$

Proof: For $n = 1$, the proof follows from Theorem 2.5. If $n = 2$, the graph has order and size 4, it can't be packable by a single tree. For, in the bijection (X, Y), the vertices u_1, u_2 of X is adjacent to v_1, v_2 of Y form a cycle $u_1\, v_2\, u_2\, v_1\, u_1$.
Hence, $w_l\,(edt\,K_{2,2}) = 2$.

The graph $K_{n, n}$ ($n \geq 3$) has order $2n$ and size n^2. In the partition (X, Y) the n vertices of X are jointed to n vertices of Y. The end vertex of X joins n vertices of Y and each vertex of Y is jointed to corresponding vertices of X gives a tree of size $n + n - 1$ and it is the largest tree of size $2n - 1$. The next vertex of X joins ($n - 1$) vertices of Y and the end vertex of the corresponding side of Y joins ($n - 1$) vertices of X, gives a tree of size $2n - 3$, for one edge is common in both case. Proceeding like this, ($n - 1$) th vertex of X has only two vertices of Y to join and ($n - 2$) th vertex of Y joins two vertices of X gives a tree of size three. The other end vertex of X joins no vertex of Y except the ($n - 1$) th one. But we cannot combine any of these trees except the last two. Hence, the complete bipartite graph $K_{n, n}$ ($n \geq 3$), $w_l\,(edt\,K_{n,\,n}) = n - 1$ with e ($edt\ T_i$) are $2n - 1$, $2n - 3$, ...7, 5, 4. This completes the proof.

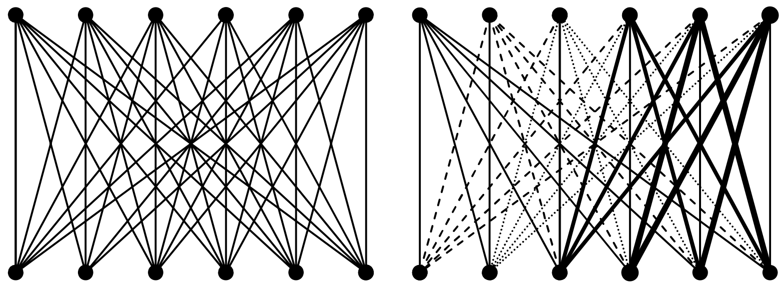

Figure 4: *$K_{n,n}$ (n = 6), 6 – 1 = 5 edge disjoint trees of size 11, 9, 7, 5, 4*

Theorem 2.7: *Let G is k- regular. Then*

$$w_l(edt\ G) = \begin{cases} k, & \text{if } k \le 2 \\ k-1, & \text{if } k \ge 3 \end{cases}$$

Proof: For $k = 1$, G is the graph with isolated vertex or P_2 and the theorem is true. For $k = 2$, it follows from Theorem 2.1. Since, $d_G(v) = k$, ($k \geq 3$), G must be packable by a minimum of $k - 1$ mutually edge disjoint trees. Therefore, w (edt $G) \geq k - 1$. This proves the Theorem.

Theorem 2.8: *For the connected graph G, the minimum number of edge disjoint tree, w_l (edt G) = n if and only if G is isomorphic to any one of the following: K_{n+2}, $K_{n+1, n+1}$, (k + 1) – regular graph, or a graph with $\delta(G) = n$, ($n \geq 3$).*

Proof: The proof follows from Theorem 2.3, 2.5, 2.6 and 2.7.

3. Observations:

Observation 3.1: Helm graph H_n (n ≥ 3) is the packing of at least 2 edge disjoint trees.

Observation 3.2: Wheel graph W_n (n ≥ 4) is the packing of at least 2 edge disjoint trees.

Observation 3.3: Gear graph is the packing of at least 2 edge disjoint trees.

Observation 3.4: Herschel graph is the packing of at least 2 edge disjoint trees.

Observation 3.5: Petersen graph is the packing of at least 2 edge disjoint trees.

Observation 3.6: Tetrahedron graph is the packing of at least 2 edge disjoint trees.

Observation 3.7: Octahedron graph is the packing of at least 3 edge disjoint trees.

Observation 3.8: Icosahedrons graph is the packing of at least 4 edge disjoint trees.

Observation 3.9: The maximum edge length of a tree is 2n – 1 in complete bipartite graph $K_{n, n}$ $(n \geq 2)$.

Observation 3.10: The maximum edge length of a tree is n – 1 in complete graph K_n

4. Open Question

Find the minimum number of edge disjoint trees that a k – tree should be packable.

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