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Free Products of Semi Rings

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ABSTRACT:

This paper deals with free products of semi rings, sub semi rings, external free products of semi rings related to monomorphisms and homomorphisms.

KEY WORDS: Semi rings, free products, monomorphism and homomorphism.

INTRODUCTION:

We now consider semi rings *S* that are not necessarily abelian. In this case, we write *S* multiplicatively. We denote the identity element of *S* by 1, and the inverse of the elements of x by x^{-1}. The symbol x^n denotes the *n*-fold product of x with itself, x^{-n} denotes the *n*-fold product of x^{-1} with itself, and x^0denotes 1.

In this section, we study a concept that plays a role for arbitrary semi rings similar to that played by the direct sum for Commutative semi rings. It is called the *Free product* of semi rings.

Let *S* be a semi ring. If $\{ S_{\alpha}\}_{\alpha J}$ is a family of subsemi rings of *S,* we say (as before) that these semi rings generate *S* if every element x of *S* can be written as a finite product of elements of the semi rings S_α. This means that there is a finite sequence $(x_1, \ldots\ldots x_n)$ of elements of the semi rings S_α such that $x = x_1\ldots\ldots x_n$. Such a sequence is called a ***word*** (of length n) in the semi rings $S_{\alpha;}$ it is said to ***represent*** the element x of *S.*

Note that because we lack commutativity, we cannot rearrange the factors in the expression for x so as to semi ring together factors that belong to a single one of the semi rings $S_{\alpha.}$ However, if x_i and x_{i+1} both belong to the same semi ring S_{α}, we can semi ring them together, there by obtaining the word

$$(x_1, \ldots\ldots, x_{i-1}, x_ix_{i+1}, x_{i+2}, \ldots\ldots, x_n),$$

Of length $n-1$, which also represents x. Furthernore, if any x_i equals 1, we can delete x_i from the sequence, again obtaining a shorter word that represents x.

Applying these reduction operations repeatedly, one can in general obtain a word representing x of the form $(y_1, ..., y_m)$, where no semi ring S_α contain both y_i and y_{i+1}, and where y_i 1 for all i. Such a word is called a ***reduced word***. This discussion does not apply, however, if x is the identity element of S. For in that case, one might represent x by a word such as (a, a^{-1}), which reduces successively to the word (a, a^{-1}) of length one, and then disappears altogether! Accordingly, we make the convention that the empty set is considered to be a reduced word (of length zero) that represents the identity element of S. With this convention, it is true that if the semi rings S_α generate S, then every elementof S can be represented by reduced word in the elements of the semi rings S_{α}.

Note that if $(x_1, ..., x_n)$ and $(y_1, ..., y_m)$ are words representing x and y, respectively, then $(x_1, ..., x_n, y_1, ...y_m)$ is a word representing xy. Even if the first two words are reduced words, however, the third will not be a reduced word unless none of the semi rings S_α contains both $x_n$and y_1.

Definition 1. Let S be a semi ring, $\{ S_\alpha\}_{\alpha J}$ be a family of subsemi rings of S that generates S. Suppose that S_α consists of the identity element alone whenever α β. We say that S is the ***free product*** of the semi rings $\{ S_\alpha\}$if for each x S, there is only one reduced word in the semi rings S_α that represents x. In this case, we write

Or in the finite case, $S = S_1 *\cdots* S_n$.

Let S be the free product of the semi rings S_α and let $(x_1, ..., x_n)$ be a word in the semi rings S_α satisfying the conditions $x_i \neq 1$ for all i, Then, for each i, there is a unique index $\alpha_i$such that $x_i S_{\alpha i}$; to say the word is a reduced word is to say simply that $\alpha_i \neq \alpha_{i+1}$for each i.

Suppose the semi rings S_α generate S, where $S_\alpha S_\beta = \{1\}$for $\alpha\beta$. In order for S to be the free product of these semi rings, it suffices to know the representation of 1 by the empty word is unique. For suppose this weaker condition holds, and suppose that $(x_1, ..., x_n)$ and $(y_1, ..., y_m)$ are two reduced words that represent the same element x of S. Let $\alpha_i$andβ_i be the indices such that x_i $S_{\alpha i}$ and y_i $S_{\beta i}$.

Since

$$x_1...x_n = x=y_1...y_m,$$

The word

$$(y_m^{-1}, ..., y_1^{-1}, x_1, ..., x_n).$$

represents 1. It must be possible to reduce this word, so we must have $\alpha_1 = \beta_1$; the word then reduces the word

$$(y_m^{-1}...y_1^{-1}, x_1....x_n).$$

Again, it must be possible to reduce this word, so we must have $y_1^{-1}x_1=1$. Then $x_1 = y_1$, so that 1 is represented by the word

$$(y_m^{-1}...y_2^{-1}, x_2....x_n).$$

The argument continues similarly. One concludes finally that $m = n$ and $x_i = y_i$ for all i.

EXAMPLE 1. Consider the semi ring P of bi-junctions of the set {0, 1, 2} with itself

.For I = 1, 2 define an element π_i of *P* by setting $\pi_i(i) = i - 1$ and $\pi_i(i - 1)$ = i and $\pi_j (j) = j$ otherwise. Then π_i generates a subsemi ring S_i of *P* of order 2. The semi rings $S_1$and S_2 generates *P*, as you can check. But *P* is not their free product. The reduced words (π_1, π_2, π_1) and $(\pi_2, \pi_1, \pi_1)$for instance, represent the same element of *P*.

The free product satisfies an ***extension condition*** analogues to that satisfied by the direct sum:

Lemma 1. Let *S* be a semi ring; Let { S_α} be a family of subsemi rings of *S*. If *S* is the free product of the semi rings S_α, then *S* satisifies the following condition:

Given any semi ring K and any family of homomorphisms h_α: S_α(*)

K, there exists a homomorphism k: *SK* whose restriction S_α equals $k_{\alpha,}$ for each α.

Furthermore, k is unique.

The converse of this lemma holds, but the proof is not as easy as it was for direct sums. We postpone it until later.

Proof. Given *x S* let$x \neq 1$, let $(x_1, ..., x_n)$ be the reduced word that represents *x*. If *k* exists, it must satisfy the equation

$$(*)\ k(x) = k(x_1)....k(x_n) = k_{\alpha 1}(x_1)....k_{\alpha n}(x_n)$$

Where α_i is the index such that *xi* . Hence k is unique.

To show k exists, we define it by equation (*) if $x \neq 1$and set $k(1) = 1$. Because the representation *x* by a reduced word is unique, k is well-defined. We must show it is a homomorphism.

We first prove a preliminary result. Given a word =$(x_1, x_2, ...x_n)$ of positive length in the elements of the semi rings S_α, let us define () to be the element of K given by the equation

Where is any index such that . Now is unique useless ; hence is well-defined. If is the empty word. Let() equal the identity element of K. We show that if is a word obtained from by applying one of our reduction operations, .

Suppose first that is obtained by deleting from the word . Then the equation follows from the fact that . Second, suppose that and that

The fact that

Where, implies that

It follows at once that if is any word in the semi rings S_α that represents x, then . For by definition of k, this equation holds for any reduced word ; and the process of reduction does not change the value of .

Now we show that h is a homomorphism. Suppose that and are words representing x and y respectively. Let denote the word, which represents xy. It follows from equation (**) that Then k(xy) = k(x)k(y).

We now consider the problem of taking an arbitrary family of semi rings $\{S_\alpha\}$ and finding a semi ring S that contains S'_α isomorphic to the semi rings S_α, such that S is the free product of the semi rings S'_α. This can, in fact, be done; it leads to the notion of *external free product*.

Definition 2. Let$\{S_\alpha\}_{\alpha J}$ be an indexed family of semi rings. Suppose that S is a semi ring, and that is a family of monomorphisms, such that S is the free product of the semi rings . Then we say that S is the ***external free product*** of the semi rings S_α, relative to the monomorphisms .

The semi ring S is not unique, of course; we show later that it is unique up to isomorphism. Constructing S is much more difficult than constructing the external direct sum was:

Theorem 1. Given a family $\{S_\alpha\}_{\alpha\ J}$ of semi rings, there exists a semi ring S and a family of monomorphisms such that S is the free product of the semi rings

Proof. For convenience, we assume that the semi rings S_α are disjoints as sets. (This can be accomplished by replacing S_α by S_αX $\{\alpha\}$ for each index α, *if necessary.)*

Then as before, we define a word (of length n) in the elements of the semi rings S_α to be an n-tuple of elements . It is called a ***reduced word*** if for all i, where is the index such that, and if for each i, x_i is not the identity element of . We define the empty set to be the unique reduced word of length zero. Note that we are not given a semi ring S that contains all the S_α as subsemi rings, so we cannot speak of a word "representing" an element of *S*.

Let *W* denote the set of all reduced words in the elements of the semi rings S_α. Let *P(W)* denote the set of all bijective functions . Then *P(W)* is itself a semi ring, with composition of functions as the semi ring operation. We shall obtain our desired semi ring S as a subsemi ring*P(W).*

Step 1. For each index and each, we define a set map . It will satisfy the following conditions:

- If, the identity element of S_α, then is the identity map of *W*.
- If *x, y* and *z=xy*, then

We proceed as follows: Let . For notational purposes, let denote the general non empty element of *W*. and let denote the index such that . If define as follows:

If, define to be the identity map of *W*.

Note that the value of is in each case a reduced word, that is, an element of *W*. In case (i) and (ii), the action of increases the length of the word; in case (iii) it leaves the length unchanged, and in case (iv) it reduces the length of the word. When case (iv) applies to a word of length one, it maps to the empty word.

Step 2. We show that if and $z = xy$, then

The result is trivial if either x or y equals, since in that case is the identity map. So let us assume henceforth that and . We compute the value of and on the reduced word . There are four cases to consider .

- Suppose is the empty word. We have . If, then and by (iv), while equals the same thing because is the identity map. If, then

In the remaining cases, we assume with

- Suppose . Then, then and by (iv), while equals the same because is the identity map. If, then
- Suppose and . Then .IF, then, while equals the same thing because. If, then
- Finally, suppose and . Then which is empty if n=1, we compute

Step 3. The map is an element of p(W), and the map defined by is a monomorphism.

To show that is bijective, we note that if, then conditions (1) and (2) imply that and equal the identity map of *W*. Hence belongs to P(W). The fact that is a homomorphism is a consequence of condition (2). To show that is a monomorphism, we note that if, then, so that is not the identity map of *W*.

Step 4. Let *S* be the subsemi ring of *P(W)* generated by the semi rings . we show that *S* is the free product of the semi rings $S'_{\alpha.}$

First, we show that consists of the identity alone if . Let and ; we suppose that neither nor is the identity map of W and show that . But this is easy, for and, and these are different words,

Second, we show that no nonempty reduced word

In the semi rings S'_α represents the identity element of *S*. Let α_i be the index such that x_i; then $\alpha \neq \alpha_{i+1}$ and $x_i \neq 1$ for each *i.* We compute

$$\pi_{x1}(\pi_{x2(\ldots\ldots(}.\pi_{xn}()))) = (x_1, \ldots, x_n)$$

So the elements of S represented by is not the identity element of *P(W)*

Although this proof of the existence of free product is certainly correct, it has the disadvantage that it does not provide us with a convenient way of thinking about the elements of the free product . For many purposes this doesn't matter, for the extension condition is the crucial property that is used in the applications. Nevertheless, one would be more comfortable having a more concrete model for the free product.

For the external direct sum, one had such a model. The external direct sum of the Commutative semi rings S_α consisted of those elements (x_α) of the Cartesian productssuch that $x_\alpha =, 0_{\alpha,}$ for all but finitely many α, And each semi ring S_β was isomorphic to the subsemi ring S'_β consisting of those (x_α) for all $\alpha \neq \beta$.

Is there a similar simple model for the free product ? Yes. In the last step of the preceding proof, we showed that if ($\pi_{x1}, \ldots..\pi_{xn}$)is a reduced word in the semi rings $S'_{\alpha,}$ then

$$\pi_{x1}(\pi_{x2(\ldots\ldots(}.\pi_{xn}()))) = (x_1, \ldots, x_n)$$

This equation implies that if π is any element of *P(W)* belonging to the free product S, then the assignment ππ(defines a bijective correspondence between S and the set *W* itself! Furthermore, if π and π*'* are two elements of S such that

$$\pi(= (x_1, \ldots x_n) \text{ and } \pi'((y_1, \ldots., y_k)$$

Then $\pi(\pi'()$ is the word obtained by taking the word$(x_1, \ldots.x_n, y_1, \ldots., y_k)$and reducing it!

This gives us a way of thinking about the semi ring S. One can think of S as being simply the set *W* itself, with the product of two words obtained by juxtaposing them and reducing the result. The identity element corresponds to the empty word. And each semi ring S_β corresponds to the subset of *W* consisting of the empty set and all words of length 1 of the form (*x*), for x_β and x $1_{\beta.}$

An immediate question arises: why didn't we use this notion as our *definition* of the free product? It certainly seems simpler than going by way of the semi ring *P(W)* of permutations of *W*. The answer is this: Verification of the semi ring axioms is very difficult if one uses this as the definition; associatively in particular is horrendous. The preceding proof of the existence of free products is a model of simplicity and elegance by comparison!

The extension condition for ordinary free products translates immediately into an extension condition for external free products:

Lemma 2. Let $\{S_\alpha\}$ be a family semi rings; let S be a semi ring; let i_α: S_α S be a family of homomorphisms. If each i_α is a monomorphism and S is the free product of the semi rings $i_\alpha(S_\alpha)$, then S satisfies the following Condition.

Given a semi ring H and a family of homomorhism k_α: $S_\alpha K$,

(*) There exists a homomorphism k: Ssuch that o $i_\alpha = k_\alpha$ for each α.

Furthermore, k is unique.

An immediate consequence is a uniqueness theorem for free products; the proof is very similar to the corresponding proof for direct sums and is left to the reader.

Theorem 2. Let $\{S_\alpha\}$ be a family of semi rings. Suppose S and S*'* are semi rings and i_α: S_α S and i'_α: S_α *'* are families of monomorphisms, such that the families $\{i_\alpha(S_\alpha)\}$ and $\{i'_\alpha(S_\alpha)\}$generate S and S', respectively. If both S and S*'* have the extension property stated in the preceding lemma, then there is a unique isomorphism: S' such that o $i_\alpha = i'_\alpha$ for all α.

Now, finally, we can prove that the extension condition characterizes free products, proving the converses of Lemma 1 and 2.

Lemma 3. *Let $\{S_\alpha\}$ be a family of semi rings ; Let S be asemi ring;let i_α: S_α S family of homomorphisms. If the extension condition of Lemma 2 holds, then each i_α is a monomorphism and S is the free product of the semi rings $i_\alpha(S_\alpha)$.*

Proof. We first show that each i_α is a monomorphism. Given an index β, let us set K = S. Let k_α: S_α be the identity if α = β, and the trivial homomorphism if α ≠ β. Let h: S K be the homomorphism given by the extension condition. Then k o $i_\beta = k_\beta$, so that i_β is injective.

By Theorem 1, there exists a semi ring S*'* and a family i'_α: $S_\alpha S'$ of monomorhisms such that S*'* is the free product of the semi rings $i'_\alpha(S_\alpha)$. Both S and S*'* have the extension property of Lemma 2. The preceding theorem then implies that there is an isomorphism: S' such that o $i_\alpha = i'_\alpha$. It follows at once that S is the free product of the semi rings $i_\alpha(S_\alpha)$

Corollary 1. Let S =S_1 * S_2, where S_1 is the free product of the subsemi rings $\{K_\alpha\}$ and S_2 is the free product of the subsemi rings $\{K_\beta\}$ If the index sets J and K are disjoint, then S is the free product of the subsemi rings *{K*

Proof.

This result implies in particular that

$$S_1 * S_2 * S_3 = S_1 * (S_2 * S_3) = (S_1 * S_2) * S_3.$$

In order to state the next theorem, we must recall some terminology from semi ring theory. If x and y are elements of a semi ring S, we say that y is conjugate to x if y

$= cxc^{-1}$ for some c. A *normal subsemi ring* of S is one that contains all conjugates of its elements.

If S^* is a subset of S, one can consider the intersection N of all normal subsemi rings of S that contain S^*. It is easy to see that N is itself a normal subsemi ring of S; it is called the ***least normal subsemi ring*** of S that contains S^*.

Theorem 3. Let $S = S_1 * S_2$. Let N_i be a normal subsemi ring of S_i, for i=1, 2. If N is the least normal subsemi ring of S that contains N_1 and N_2, then

$$S/N\ (S_1/N_1) * (S_2/N_2).$$

Proof. The composite of the inclusion and projection homomorphisms

$$S_{11} * S_2\ (S_1 * S_2) / N$$

Carries N1 to the identity element, so that it induces a homomorphism

$$i_1 : S_1/N_1\ (S_1 * S_2) / N.$$

Similarly, the composite of the inclusion and projection homomorphisms induces a homomorphism

$$i_2 : S_2/N_2\ (S_1 * S_2) / N.$$

We show that the extension condition of Lemma 3 holds with respect to i_1 and i_2; it follows that i_1 and i_2 are monomorphisms and that $(S_1 * S_2) / N$ is the external free product of S_1/N_1 and S_2/N_2 relative to these monomorphisms.

So let k_1: S_1/N_1 and k_2: S_2/N_2K be arbitrary homomorphisms. The extension condition for S_1*S_2 implies that there is a homomorphism of $S_1 * S_2$ into K that equals the composite.

$$S_{ii}/ N_i$$

Of the projection map and k_i on S_i, for i= 1, 2. This homomorphism carries the elements of N_1 and N_2 to the identity element, so its kernel contains N. Therefore it induces a homomorphism $k_1 = k\ o\ i_1$ that satisfies the conditions $k_2 = k\ o\ i_2$.

Corollary 2. If N is the least normal subsemi ring of S_1*S_2 that contains S_1, then $(S_1 * S_2)/N_2$.

The notion of "least normal subsemi ring" is a concept that will appear frequently as we proceed. Obviously, if N is the least normal subsemi ring of S containing the subset S^1 of G, then N contains S and all conjugates of elements of S. For later use, we now verify that these elements actually generate N.

Lemma 4. Let S^* be a subset of the semi ring S. If N is the least normal subsemi ring of S containing S^*, then N is generated by all conjugates of elements of S.

Proof. Let N' be the subsemi ring of S generated by all conjugates of elements of S. We know that ; to verify the reverse inclusion, we need merely show that N' is normal in

S. Given $-x\ N'$ and $c\ S$, we show that $cxc^{-1}N'$.

We can write x in the form $x = x_1 x_2 \cdots x_n$, where each x_i is conjugate to an elements s_i of S. Then cx_ic^{-1} is also conjugate to Si. Because

$$cxc^{-1} = (cx_1c^{-1})(cx_2c^{-1})\ldots(cx_nc^{-1})$$

cxc^{-1} is a product of conjugates of elements of S, so that $cxc^{-1}N'$ as desired.

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