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Some Coupled Fixed Point Theorems in Partially Ordered Metric Space

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Abstract

Resently, J. Harjani, B. Lopez and K. Sadarangani establi established for mappings satisfying a rational type contractive condition in partially ordered metric space. In this paper, we obtain some corresponding coupled fixed point theorems in partially ordered metric spaces by employing a rational type contractive condition.

Keywords: Coupled fixed point theorems; partially ordered metric spaces; rational type contractive condition.

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1. Introduction

The notion of coupled fixed points was introduced by Chang and Ma [3]. Since then, the concept has been of interest to many researchers in metrical fixed point theory. Bhaskar and Lakshmikantham [2] established coupled fixed point theorems in a metric space endowed with partial order by employing the following contractivity condition: For a mapping $T: X \times X \to X$, there exists $k \in (0, 1)$ such that $d\left(T(x,y), T(u,v)\right) \leq \frac{k}{2}[d(x,u)+d(y,v)] \forall x,y,u,v \in X, x \geq u, y \leq v$. Harjani et al [5] estabished some fixed point theorem in partially ordered metric space setting by using a contractive condition of rational type. In this paper, we shall prove corresponding coupled fixed point theorems in partially ordered metric space by employing some notions of Bhaskar and Lakshmikantham [2] as well as a rational type contractive condition.

2. Preliminaries

Definition 2.1: Let (X, d) be a metric space. An element (x, y) ∈ X x X is said to be a coupled fixed point of the mapping T: X x X → X if T(x, y) = x and T(y, x) = y.
Definition 2.2: Let (X, ≤) be a partially ordered set and T: X x X → X. We say that T has the mixed monotone property if T(x, y) is monotone nondecreasing in x and monotone nonincreasing in y, that is for all x, y ∈ X,

$$\forall\ x_1, x_2 \in X\ ,\ x_1 \le x_2 \Rightarrow T(x_1, y) \le T\ (x_2, y)$$

$$\forall\ y_1, y_2 \in X\ ,\ y_1 \le y_2 \Rightarrow T(x, y_1) \ge T\ (x, y_2)$$

3. Main results

Let (X, ≤) be a partially ordered metric set and d be a metric on X such that (X, d) is a complete metric space. We also endow the product space X x X with the following partial order: for (x, y), (u, v) ∈ X x X, (u, v) ≤ (x, y) ⇔ x ≥ u, y ≤ v.

Theorem 3.1: Let (X, ≤) be a partially ordered metric set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let T: X x X → X be a continuous mapping which has the mixed monotone property such that,

for some $\alpha, \beta \in [0,1)$, $\forall x, y, u, v \in X, x \neq u$, *we have*

$$d(T(x,y),T(u,v)) \le \alpha\left(\frac{d(x,T(x,y)) + d(u,T(u,v))}{d(x,u) + d(x,T(x,y))}\right) d(x,T(x,y)) + \beta d(x,u),\ \alpha + \beta < 1. \qquad (1)$$

Then T has a coupled fixed point.

Proof. Choose $(x_0, y_0) \in X \times X$ and set $x_1 = T(x_0, y_0)$, $y_1 = T(y_0, x_0)$, and in general, $x_{n+1} = T(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$.

With $x_0 \le T(x_0, y_0) = x_1$ (say) and $y_0 \ge T(y_0, x_0) = y_1$ (say). By the iterative process above, $x_2 = T(x_1, y_1)$ and $y_2 = T(y_1, x_1)$. Therefore,

$T^2(x_0, y_0) = T(T(x_0, y_0), T(y_0, x_0)) = T(x_1, y_1) = x_2$,
and $T^2(y_0, x_0) = T(T(y_0, x_0), T(x_0, y_0)) = T(y_1, x_1) = y_2$.

Due to the mixed monotone property of T, we obtain

$x_2=T^2(x_0, y_0) = T(x_1, y_1) \ge T(x_0, y_0)=x_1$, $y_2=T^2(y_0, x_0) = T(y_1, x_1) \le T(y_0, x_0) = y_1$.

In general, we have that for n ϵ N,

$x_{n+1} = T^{n+1}(x_0, y_0)=T(T^n(x_0, y_0), T^n(y_0, x_0))$, $y_{n+1} = T^{n+1}(y_0, x_0) =T(T^n(y_0, x_0), T^n(x_0, y_0))$

It is obvious that

$x_0 \le T(x_0, y_0) = x_1 \le T^2(x_0, y_0) = x_2 \le \ldots\ldots\ldots \le T^n(x_0, y_0) = x_n \le \ldots\ldots\ldots\ldots$,
and $y_0 \ge T(y_0, x_0) = y_1 \ge T^2(y_0, x_0) = y_2 \ge \ldots\ldots\ldots \ge T^n(y_0, x_0) = y_n \ge \ldots\ldots\ldots\ldots$,

Therefore, we have by condition (1) that

$$d(x_{n+1}, x_n) = d\ (T(x_n, y_n), T\ (x_{n-1}, y_{n-1})\ )$$

$$\le \alpha\left(\frac{d(x_n, T(x_n, y_n)) + d(x_{n-1}, T(x_{n-1}, y_{n-1}))}{d(x_n, x_{n-1}) + d(x_n, T(x_n, y_n))}\right) d(x_n, T(x_n, y_n)) + \beta d(x_n, x_{n-1})$$

$$= \alpha\left(\frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{d(x_n, x_{n-1}) + d(x_n, x_{n+1})}\right) d(x_n, x_{n+1}) + \beta\, d(x_n, x_{n-1})$$

$= \alpha\ d(x_n, x_{n+1}) + \beta\ d(x_n, x_{n-1})$

from which it follows that

$$d(x_n, x_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right) d(x_n, x_{n-1}) \text{ ....(2)}$$

$d(y_{n+1}, y_n) = d\,(T(y_n, x_n), T\,(y_{n-1}, x_{n-1}))$

$$\leq \alpha\left(\frac{d(y_n, T(y_n, x_n))\ d(y_{n-1}, T(y_{n-1}, x_{n-1}))}{d(y_n, y_{n-1}) + d(y_n, T(y_n, x_n))}\right) d(y_n, T(y_n, x_n)) + \beta\, d(y_n, y_{n-1})$$

$$= \alpha\left(\frac{d(y_n, y_{n+1}) + d(y_{n-1}, y_n)}{d(y_n, y_{n-1}) + d(y_n, y_{n+1})}\right) d(y_n, y_{n+1}) + \beta\, d(y_n, y_{n-1})$$

$= \alpha\ d(y_n, y_{n+1}) + \beta\ d(y_n, y_{n-1})$

$$d(y_n, y_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right) d(y_n, y_{n-1}) \text{ ....(3)}$$

From (2) and (3),

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right)(d(x_n, x_{n-1}) + d(y_n, y_{n-1})) \text{..........(4)}$$

Let $\delta_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$ and $\lambda = \frac{\beta}{1-\alpha}$. Then, we have from (4) that

$\delta_n \leq \lambda\delta_{n-1} \leq \lambda^2\delta_{n-2} \leq \ldots\ldots\ldots\ldots \leq \lambda^n\delta_0.$ ....(5)

If $\delta_0 = 0$, then (x_0, y_0) is a coupled fixed point of T.

Suppose that $\delta_0 > 0$. Then, for each $r \in N$, we obtain by (5) and the repeated application of triangle inequality that

$d(x_n, x_{n+r}) + d(y_n, y_{n+r}) \leq [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+r-1}, x_{n+r})]$
$+[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+r-1}, y_{n+r})]$
$= [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] + [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})]$
$+ \cdots + [d(x_{n+r-1}, x_{n+r}) + d(y_{n+r-1}, y_{n+r})]$
$\leq \delta_n + \delta_{n+1} + \cdots + \delta_{n+r-1}$

$$\leq \frac{\lambda^n (1-\lambda^r)\delta_0}{1-\lambda} \to 0 \text{ as } n \to \infty \text{ .....(6)}$$

Therefore, $\{x_n\}$, $\{y_n\}$ are Cauchy sequences in (X, d).

Since (X, d) is a complete metric space, there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. We now show that (x^*, y^*) is a coupled fixed point of T.Let $\epsilon > 0$. Continuity of T at (x^*, y^*) implies that, for a given $\epsilon/2 > 0$, there exists a $\delta > 0$, such that $d(x^*, u) + d(y^*, v) < \delta$ implies $d(T(x^*, y^*), T(u, v)) < \epsilon/2$.

Since $\{x_n\} \to x$ and $\{y_n\} \to y$, for $\zeta = \min(\epsilon/2, \delta/2) > 0$, there exist n_0, m_0, such that, for $n \geq n_0$, $m \geq m_0$, we have $d(x_n, x^*) < \zeta$, and $d(x_m, x^*) < \zeta$.

Therefore, for $n \in IN$, $n \geq \max\{n_0, m_0\}$,

$d(T(x^*, y^*), x^*) \leq d(T(x^*, y^*), x_{n+1}) + d(x_{n+1}, x^*)$
$= d(T(x^*, y^*), T(x_n, y_n)) + d(x_{n+1}, x^*) < \epsilon/2 + \zeta \leq \epsilon,$

from which it follows that T (x*, y*) = x*. In a similar manner, we can show that T (y*, x*) = y*.

Hence, (x*, y*) is a coupled fixed point of T.

This complete the proof.

Theorem 3.2: In Theorem3.1, Adding the condition that there exists z ∈ X which is comparable to x and y, ∀ x, y ∈ X. Then, T has a unique coupled fixed point.

Suppose that there exist (x*, y*), (x′, y′) ∈ X × X are coupled fixed points of T.

Case(I): If x*, x′ are comparable and y*, y′ are also comparable, and x* ≠ x′, y* ≠ y′ then by the contractive condition, we have

d (x*, x′) = d (T(x*, y*), T (x′, y′))

$$\leq \alpha\left(\frac{d(x^*,T(x^*,y^*))+d(x',T(x'y'))}{d(x^*,x')+d(x^*,T(x^*,y^*))}\right)d(x^*,T(x^*,y^*))+\beta d(x^*,x')$$

$$=\alpha\left(\frac{d(x^*,x^*)+d(x',x')}{d(x^*,x')+d(x^*,x^*)}\right)d(x^*,x^*)+\beta d(x^*,x')=\beta d(x^*,x')$$

which gives d(x*, x′)≤ 0, β < 1 (a contradiction). Thus, x* = x′.

d (y*, y′) = d (T(y*, x*), T (y′, x′))

$$\leq \alpha\left(\frac{d(y^*,T(y^*,x^*))+d(y',T(y',x'))}{d(y^*,y')+d(y^*,T(y^*,x^*))}\right)d(y^*,T(y^*,x^*))+\beta d(y^*,y')$$

$$=\alpha\left(\frac{d(y^*,y^*)+d(y',y')}{d(y^*,y')+d(y^*,y^*)}\right)d(y^*,y^*)+\beta d(y^*,y')=\beta d(y^*,y')$$

which gives d(y*, y′) ≤ 0, (a contradiction).

Hence, y* = y′. Therefore, (x*, y*) is a unique coupled fixed point of T.

Case II: If x* is not comparable to x′ and y* is not comparable to y′ then by the contractive condition, there exists w comparable to x* and x′ and there exists v comparable to y*and y′.

Monotonicity implies that w_n is comparable to $x_n^* = T\left(x_{n-1}^*, y_{n-1}^*\right) = x^*$, and w_n is comparable to w_1. Also, monotonicity implies that y_n^* is comparable to v and y_n^* is also comparable to w_2.

On the other hand, if $x_n^* \neq w_1$, $x_n' \neq w_1$, then by the contractive condition, we get

$$d\left(w_1, x_n^*\right)=d\left(T(w_1,w_2), T\left(x_{n-1}^*, y_{n-1}^*\right)\right)$$

Case III: If (x*, y*) is not comparable to (x′, y′), then there exists (w, v) comparable to (x*, y*) and (x′, y′). Monotonicity implies that

$(T^n(w, v), T^n(v, w))$

$$d\left(\begin{pmatrix} x^* \\ y^* \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \right) = d\left(\begin{pmatrix} T^n(x^*, y^*) \\ T^n(y^*, x^*) \end{pmatrix}, \begin{pmatrix} T^n(x', y') \\ T^n(y', x') \end{pmatrix} \right)$$

$$\le d\left(\begin{pmatrix} T^n(x^*, y^*) \\ T^n(y^*, x^*) \end{pmatrix}, \begin{pmatrix} T^n(w, v) \\ T^n(v, w) \end{pmatrix} \right) + d\left(\begin{pmatrix} T^n(w, v) \\ T^n(v, w) \end{pmatrix}, \begin{pmatrix} T^n(x', y') \\ T^n(y', x') \end{pmatrix} \right)$$

$$\le d\left(T^n(x^*, y^*), T^n(w,v)\right) + d\left(T^n(y^*, x^*), T^n(v, w)\right)$$

$$+ d\left(T^n(w,v), T^n(x', y')\right) + d\left(T^n(v, w), T^n(y', x')\right)$$

$$\le \alpha^n \left(\frac{d\left(x^*, T^n(x^*, y^*)\right) + d\left(w, T^n(w,v)\right)}{d(x^*, w) + d\left(x^*, T^n(x^*, y^*)\right)} \right) d\left(x^*, T^n(x^*, y^*)\right) + \beta^n d(x^*, w)$$

$$+ \alpha^n \left(\frac{d\left(y^*, T^n(y^*, x^*)\right) + d\left(v, T^n(v, w)\right)}{d(y^*, v) + d\left(y^*, T^n(y^*, x^*)\right)} \right) d\left(y^*, T^n(y^*, x^*)\right) + \beta^n d(y^*, v)$$

$$+ \alpha^n \left(\frac{d\left(w, T^n(w, v)\right) + d\left(x', T^n(x', y')\right)}{d(w, x') + d\left(w, T^n(w, v)\right)} \right) d\left(w, T^n(w, v)\right) + \beta^n d(w, x')$$

$$+ \alpha^n \left(\frac{d\left(v, T^n(v, w)\right) + d\left(y', T^n(y', x')\right)}{d(v, y') + d\left(v, T^n(v, w)\right)} \right) d\left(v, T^n(v, w)\right) + \beta^n d(v, y')$$

$$= \beta^n \left[\, d(x^*, w) + d(y^*, v) + d(w, x') + d(v, y') \,\right] \to 0 \; as \; n \to \infty$$

Hence, T has a unique coupled fixed point.

References

[1] I. Beg, A. Latif, R. Ali and A. Azam; Coupled fixed point of mixed monotone operators on probabilistic Banach spaces, Archivum Math. 37 (1) (2001), 1-8.

[2] T. G. Bhaskar and V. Lakshmikantham; Fixed point theorems in partially ordered metric Spaces and applications, Nonlinear Analysis: Theory, Methods & Applications 65 (7) (2006), 1379-1393.

[3] S. S. Chang and Y. H. Ma; Coupled fixed point of mixed monotone condensing operators and existence theorem of the solution for a class of functional equations arising in dynamic programming, J. Math. Anal. Appl. 160 (1991), 468-479.

[4] L. Ciric and V. Lakshmikantham; Coupled random fixed point theorems for nonlinear contractions in partially ordered metric spaces, Stochastic Analysis and Applications 27 (2009), 1246-1259.8

[5] J. Harjani, B. Lopez and K. Sadarangani; A fixed point theorem for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Abstract and Applied Analysis, Volume (2010), Article D190701, 8pages.

[6] D. S. Jaggi; Some unique fixed point theorems, Indian Journal of Pure and Applied Mathematics 8(2) (1977), 223-230.
[7] V. Lakshmikantham and L. Ciric; Coupled fixed point theorems for nonlinear contractions in Partially ordered metric spaces, Nonlinear Analysis: Theory, Methods & Applications 70 (12) 2009), 4341-4349.
[8] J. J. Nieto and R. Rodríguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, " *Acta Mathematica Sinica*, vol. 23, no. 12, pp.2205–2212, 2007
[9] Y. Wu, "New fixed point theorems and applications of mixed monotone operator, " *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 883–893, 2008.