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Nilpotent Semirings

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ABSTRACT:

This paper contains the nilpotent semirings are characterized by centers of a semiring. So we first give some elementary results on successive centers of a semiring.

Keywords: Semiring, sub Semiring, normal semiring, normal sub semiring, nil potent semiring,

1. Introduction:

1. 1. Definition:

Let S be a semiring. Consider the center $Z(S)$ of S. We denote $Z(S)$ by $Z_1(S)$ and call it as the first center of S

We know that $Z_1(S)=(Z(S))$ is a normal subsemiring of S.

Consider the quotient semiring $S/Z_1(S)$ and this center $ZS/Z_1(S)$ Again this is a normals subsemiring of $S/Z_1(S)$ So.

$$Z(S/Z_1(S))=Z_2(S)/Z_1(S)$$

For some normal subsemiring $Z_2(S)$ of S containing $Z_1(S)$ We call $Z_2(S)$ as the second center of S.

Again consider $S/Z_2(S)$ and its center $Z(S/Z_2(S))$ which is a normal subsemiring of $S(Z_2(S))$ We obtain a normal subsemiring $Z_3(S)$ of S containing $Z_2(S)$ such that

$$Z(S/Z_2(S))=Z_3(S)/Z_2(S)$$

Here $Z_1(S)\subset Z_2(S)\subset Z_3(S)$

$Z_3(S)$ is called the third center of S.

Preceding in this way we obtain nth center $Z_n(S)$ of S which is given by

$Z_n(S)/Z_{n-1}(S) = Z(S/Z_{n-1}(S))$

The successive centers of S are such that

$Z(S) = Z_1(S) \subset Z_2(S) \subset .... \subset Z_n(S)C$

Note:
The upper found for this series of successive centers is S.

1.2 Definition : (Nilpotent Semiring):
A semiring S is said to be nil potent if $Z_m(S) = S,$ for S some positive integer m, where $Z_n(S)$ is the nth center of S, which is a normal sub semiring of G such that

$Z_n(S)/Z_{n-1}(S) = Z(S/Z_{n-1}(S)), n = 1,2..........$

The smallest positive integer m such that $Z_m(S) = S$ is called the class of nipotency of G.

1. Example:
Let S be an abelian semiring. Then $Z(S) = S \text{ or } Z_1(S) = S$

So S is nilpotent

That is, every abelian semiring is nilpotent

1. Remark: Characterization of Zn(S):
Let us find the elements of $Z_n(S)$. By the definition of $Z_n(S)$, we have that $Z_n(S)$ is a normal subsemiring of S such that

$Z_n(S)/Z_{n-1}(S) = Z(S/Z_{n-1}(S))$

Thus

$xZ_n(S) \Leftrightarrow xz_{n-1}(S) \in Z(S/Z_{n-1}(S))$

$\Leftrightarrow xZ_{n-1}(S)\,yZ_{n-1}(S) = yZ_{n-1}(S)\,xZ_{n-1}(S)$ for all $y \in S$.

$\Leftrightarrow xyZ_{n-1}(S) = yxZ_{n-1}(S),$ for all $y \in S$

$(xy)(yx)^{-1} \in Z_{n-1}(S)$ for all $y \in S$

$\Leftrightarrow xy\, x^{-1}y^{-1} \in Z_{n-1}(S)$ for all $y \in S$.

That is,

$Z_n(S) = \{x \in S / xyx^{-1}y^{-1} \in Z_{n-1}(S), \text{ for all } y \text{ in } S\}.$

1. Theorem:
A semiring of order p^r p is a prime is nilpotent. (That is, every p- semiring is nilpotent).

Proof:

Let S be a finite semiring of order p^n Where p is a prime and n is a positive integer.

Since S is a prime power order semiring, it has non – trivial center, namely $Z(S) = Z_1(S)$.

Further $|Z_1(S)|$ divides $|S|$ So that

$|Z_1(S)| = P^r$, for some integer $r, 1 \le r \le n$.

So $|S/Z(S)_1| = \frac{|S|}{Z_1(S)} = \frac{P^n}{P^r} = P^{n-r}$,

So that $S/Z_1(S)$ is also a prime power order ground and thus it has a non- trivial center, $Z(S/Z_1(S))$ And

$|ZS/S_1(S)| = p^{t_1}$ for some integer $t_1, 1 \le t_1 \le n$.

Now the second center is that normal subsemiring of S, given by

$Z_2(S) | X_1(S) = Z(S/Z_1(S))|$

So

$|Z_2(S)| Z_1(S) = |Z(S/Z_1(S))| = P^{t_1}$

That is $|Z_2(S)| = p^{t_1}.|Z_1(S)| = p^{t_1}p^2 = p^{t_{1+r}}$

$= p^{t_2}$, say, where $t_2 = t_1 + r$

So the order of $Z_2(S)$ is also a power of p, namely p^{t_2}, where $t_2 > t_1$Proceeding in this way we get that $|Z_m(S)|$ is also a power of p, say p^{t_m}

Since $Z_1(S) \subset Z_2(S) \subset \ldots\ldots\ldots\ldots \subset Z_m(S) \subset \ldots\ldots\ldots\ldots,$

$|Z_1(S)| \le |Z_2(S)| \le \ldots. \le |Z_m(S)| \le \ldots\ldots\ldots\ldots,$

And $|Z_m(S) = p^{t_m} < p^n|$, For every positive integer m and t_m is increasing as a m increases.

Since p^n is finite, we must have positive integer m such that

$|S_m(S)| = p^n$

That is $|Z_m(S)| = S$ and S is nilpotent.

1.3.Definition:

Let S be a semiring and let $Z_m(S)$ denote the mth center of S. The series $\{e\} = Z_0(S) \subset Z_1(S) \subset \ldots\ldots \subset Z_m(S) \subset \ldots\ldots$ is called the upper central series of G.

2. The connection between nilpotent semirings, normal series and solvable semirings:

2. Theorem :

A semiring S is nil potent if and only if S has a normal series.

$\{e\} = S_0 \subset S_1 \subset ..... \subset S_m = S.$

Such that $S / S_{i-1} \subset Z\left(S_i / S_{i-1}\right), 1 \leq i \leq m$.

Proof:

Let S be the nil potent semiring of class m is the least positive integer such that $Z_m(S) = S$ Where $Z_n(S)$ is the n^th^ center of S given by

$Z_n(S) / Z_{n-1}(S) = Z\left(S / Z_{n-1}(S)\right)$.

Since S is a nilpotent semiring of class m, its upper central series terminates with $Z(S)$ and is of the form.

$\{e\} = Z_0(S) \subset Z_1(S) \subset Z_2(S) \subset .... \subset Z_m(S) = S,$

Then this is a normal series of S. further for any i, 1, i m, we have

$Z_i(S) / Z_{i-1}(S) = Z\left(S / Z_{i-1}(S)\right)$

Taking S_i to be $1 \leq i \leq m$ we get the if part of the theorem.

Conversely suppose that S has a normal series

$\{e\} = S_0 \subset S_1 S_2 \subset ..... \subset S_{m-1} \subset S_m = S,$

Such that $S_i / S_{i-1} \subset Z\left(S / S_{i-1}\right), i = 1, 2, ...m$ (1)

We shall show that S is nilpotent. In fact we shall show that $Z_m(S) = S$, so that S is nilpotent.

It is given that $S_i / S_{i-1} \subset Z\left(S / S_{i-1}\right)$, for $i = 1, 2..............m$ Taking $i = 1$ this gives $S_1 / S_0 \subset Z\left(S / S_0\right)$ *or* $S_1 \subset Z(S) = Z_1(S)$, Since $S_0 = \{e\}$.

Again taking i=2, in (1) we get

$S_2 / S_1 \subset Z\left(S / S_1\right)$

Thus for any $x \in S_2, xS_1 \in S_2 / S_1$ so that $xS_1 \in Z\left(S / S_1\right)$. Hence xS_1 commutes with every element of S / S_1 That is,

$xS_1 \; yS_1 = yS_1 \; XS_1$ for all y. in G,

Or, $xy\, x^{-1} y^{-1} \in S_1$, for all $y \in S$, or,

$xy^{-1} y^{-1} \in Z_1(S)$ (Since $S_1 \, CZ_1(S)$),

For all $x \in S_2$ and $y \in S$

Recalling the characterization of $Z_m(S)$, (that is $Z_m(S)$, $= \left\{x \in Y / xyx^{-1} y^{-1} \in Z(S)_{m-1}\right\}$ For all y in S).

That is $x \in S_2$ implies that $x \in Z_2(S)$, or $S_2 \subset Z_2(S)$

Repeating this process we get $S_n \subset Z_n(S)$ for all positive integers n. Taking $n = m$ we get

$S_m \subset Z_m(S)$, or, $S \subset Z(S)$.

Trivially $Z_m(S) CS$ So $Z_m(S) = S$ and S is nil potent

1. Corollary :
Every nilpotent semiring is solvable, but not the converse.

Proof:
Let S be a nilpotent semiring. Then there exists a positive integer m such that the mth center $Z_m(S) = S$ Here for any positive integer $i, Z_i(S)$ is given by

$$Z_i\left(S_0 / Z_{i-1}(S)\right) = Z\left(S / Z_i(S)\right) \qquad (1)$$

Consider the series

$\{e\} = Z_0(S) \subset Z_1(S) \subset \ldots\ldots\ldots\ldots \subset Z_m(S) = S.$

This is a normal series since $Z_{i-1}(S) \Delta Z_i(S)$. The factors of the above series are

$Z_i(S) / Z_{i-1}(S) = Z\left(S / Z_{i-1}(S)\right)$

So every element of $Z_i(S) / Z_{i-1}(S)$ commutes with every element of $S / Z_{i-1}(S)$. Since $Z_i(S) \subset S$. This implies that every element $Z_i(S) / Z_{i-1}(S)$ commutes with every other element of $Z_i(S) / Z_{i-1}(S)$ is abelian.

That is (2) is a normal series of S with abelian factors, so that S is solvable.

The following example shows that a semirings that is solvable, need not be nilpotent. Consider S_3. We have seen that

$\{e\} \subset N = \{1, \alpha, \beta\} \subset S_3.$

Is a normal series and S_3 / N is abelian. So by the theorem 4.4.1, S is solvable.

But S_3 is not nil potent, since its first center $Z_1 = (S_3) = \{i\}$. So for no positive integer $Z_n(S_3) = S_3$ and thus S_3 is not nilpotent.

Some elementary properties of nilpotent semirings:

3. Theorem
Let S be a nilpotent semiring. Then every sub semiring of S and every homomorphic image of S are nilpotent.

Proof:
Let S be nilpotent semiring of class m, so that $Z_m(S) = S$

(i) Let H be a sub semiring of S. Then $H \cap Z(S) \subseteq Z(H)$. For if $x \in H \cap Z(S)$, then $x \in Z(S)$, so that $xs = sx$, for all g in S. Since $H \subseteq S$, $xh = hx$ for all in H. So

$x \in Z(H)$ or $H \cap Z(S) \subseteq Z(H)$.

Now $Z_2(S)/Z_1(S) = Z(S/Z(S))$.

So, if $x \in Z_2, Z[Z_1(S)]$ commutates with every element of $S/Z_1(S)$ for all y in S.

That is, for all $x \, in \, Z_2(S)$ and for all y in S, $xy\, x^{-1} y^{i-1} \in Z_1(S)$.

Hence (Since $H \subset S$) for $x \in H \cap Z_2(S)$, and for all y in H, $xy^{x-1} y^{-1} \in I \cap Z_1(S)$. So $x \in Z_2(H)$ and thus

$H \cap Z_2(S) \subseteq Z_2(H)$.

Repeating this process we get

$H \cap Z_i(S) \subseteq Z_i(H)$ for all i.

So $H = H \cap S = h \cap Z_m(S) \subset Z_m(S), or\ Z_m(S) = H$.

This shows that H is nilpotent.

(ii) Let S be nilpotent semiring and let H be a homomorphic image of S. Then there exists a positive integer m such that

$$Z_m(S) = S \qquad (1)$$

And there exists an onto homomorphism $\phi : S \to H$.

First we observe that $\phi(Z(S)) \subseteq Z(H)$. For this we have to show that $\phi(x) \varepsilon Z(H)$ for every $x \, \varepsilon Z(S)$. Then

$Xs = sx$, for all x in S.

From this we get $xs\, x^{-1} s^{-1} = e$ or, $f(xy\, x^{-1}\, y^{-1}) = \phi(e)$ the identity element of S. That is

$\phi(x)\phi(y)\phi(x)^{-1}\,\phi(y)^{-1} = \phi(e)$, for all s in S (2)

So if $h \varepsilon H$; then since f is on to, $h = \phi(s)$ for some $s \in S$ therefore

$\phi(x) h \phi(x)^{-1} h^{-1} = \phi(x)\phi(s)\phi(x)^{-1}\phi(s)^{-1} = \phi(e)$, by (2).

That is

$\phi(x) h = h\,\phi(x)$ for every h in H, so that $\phi(x) \in Z(H)$.

Hence $\phi(Z(S)) \subseteq Z(H)$ or $\phi(Z_1(S)) \subseteq Z_1(H)$

Next let $z \in Z_2(S)$. Then by the definition of $Z_2(S)$, $zy\, z^{-1} y^{-1} \in Z_1(S)$, For all z in S.

So $\phi(zy\, Z^{-1} y^{-1}) \in \phi(Z_1(S)) \geq Z_1(S) Z_1(H)$ for z in S.

That is,

$\phi(z)\phi(z)\phi(z)^{-1}\phi(y)^{-1} \in Z_1(H)$

Again by the definition of $Z,(H)$ it follows that $(z) Z_2(H)$. this shows that

$\phi(Z_2(S)) \subseteq Z_2(H)$

Proceeding in this way we get

$\phi(Z_i(S)) \subseteq Z_2(H)$ for all positive integers

Taking i=m, we get that

$\phi(Z_m(S)) \subseteq Z_m(H)$ or

$\phi(G) \subseteq Z_m(H)$ or

$H \subseteq Z_m(H)$

But trivially $Z_m(H) \subseteq H$.

So $Z_m(H) = H$ and H is nilpotent

4. Theorem

Let $H_1, H_2, ...H_m$ be a family of nil potent semirings. Then $H_1 \times H_2 \times ......, H_m$ is nilpotent.

Proof:

First let us show that $Z(H \times K) = Z(H) \times Z(K)$ For any two semirings H and K.

Since H and K are nil potent semirings there exist positive integers n and p such that

$Z_n(H) = H \text{ and } Z_n(K) = K$

First let us prove that $Z(H \times K) = Z(H) \times Z(K)$

Now$(x, y) \varepsilon Z(H \times K)$

$\Leftrightarrow (x, y)(h,k) = (h,k)(x, y)$ *for all* (h,k) *in* $H \times K$.

$\Leftrightarrow (xh, yk) = (hx, ky)$ *for all h in H and k in K.*

$\Leftrightarrow xh = hx,$ *and* $yk = ky,$ *for all h in H and k in K.*

$\Leftrightarrow x \in Z(H)$ *and* $y \in Z(K)$.

$\Leftrightarrow (x, y) \in Z(H) \times Z(K)$

Thus $Z(H \times K) = Z(H) \times Z(H)$

That is $Z_1(H\ K) = Z_1(H) \times Z_1(K)$

In the same way we can see that

$Z_i(H \times K) = Z_i(H) \times Z_i(K)$ for all positive integers i.

For $m = \max\{n, p\}$ taking i=m, we get

$z_m(H \times K) = Z_m(H) \times Z_m(K)$

$= H \times K$

(Since $Z_m(H)$=H) and $Z_m(K) = K$

So $H \times K$ is nilpotent

Using induction (or proceeding in this way) we can show that $H_1 \times H_2 \times ...... \times H_m$ is nilpotent.

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