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A note on a Common Fixed Point theorem for a pair of Weak** Commuting maps

T. Phaneendra
Applied Analysis Division,
School of Advanced Sciences,
VIT-University, Vellore-632014 (T. N.), India.
E-mail: drtp.indra@gmail.com

D. Surekha
Department of Mathematics,
Hyderabad Institute of Technology & Management,
RR District, Hyderabad (A.P.), India.
E-mail: surekhaavinash@yahoo.com

Abstract

The notion of weak** commutativity was introduced by Pathak [3] as weaker form of commutativity. Recently Gagrani [1] obtained a result for a pair of weak** commuting maps, which however is incorrectly stated. In this paper, an appropriate rectification is done in Gagrani's theorem and is restated in its correct form.

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1. Introduction

Let (X, d) denote a metric space. If f is a self-map on X, we write f^n for its nth iterate. Self-map f is idempotent if $f^2 = f$. A point $p \in X$ is a *coincidence point* of self-maps f and g if $fp = gp$, the common image $fp = gp = q$ being a point of coincidence for them. Self-maps f and g are commuting if $fg = gf$.

Definition 1.1. (Pathak, [2]) Self-maps f and g are weak* commuting if

$$d(fgx, gfx) \leq d(f^2x, g^2x) \quad \text{for all} \quad x \in X. \tag{1}$$

If $fgx = gfx$ for all $x \in X$, then $d(fgx, gfx) = 0$. Thus (1) is trivial. That is, commutativity implies weak* commutativity. However there can be a noncommuting pair which is weak* commuting (See [2]). With some additional conditions on weak* commutativity, Pathak [3] defined weak** commuting maps as follows.

Definition 1.2. (Pathak, [3]) Self-maps f and g are weak** commuting if

$$f(X) \subset g(X) \tag{2}$$

and

$$\begin{aligned} d(f^2g^2x, g^2f^2x) &\leq d(f^2gx, gf^2x) \\ &\leq d(g^2fx, fg^2x) \\ &\leq d(fgx, gfx) \\ &\leq d(f^2x, g^2x) \text{ for all } x \in X. \end{aligned} \tag{3}$$

Recently, Gagrani [1] proved the following result:

Theorem 1.3. Let f and g be self-maps on a complete metric space X satisfying the inclusion

$$f(X) \subset g(X) \tag{4}$$

and the inequality

$$\begin{aligned} d(f^2g^2x, g^2f^2y) \leq &\frac{a[d(g^2x, f^2g^2x)d(g^2x, g^2f^2y) + d(f^2y, g^2f^2y)d(f^2y, f^2g^2x)]}{d(g^2x, g^2f^2y) + d(f^2x, f^2g^2x)} \\ &+ bd(g^2x, f^2y) \quad \text{for all} \quad x, y \in X, \end{aligned} \tag{5}$$

where $a, b \geq 0$ such that $a + b < 1$. If g is continuous and (f, g) is weak** commuting, then f and g will have a unique common fixed point.

We point out that Theorem 1.3 breaks down if the donominator on the right hand side of (5) is zero. In fact, if

$$d(g^2x, g^2f^2y) + d(f^2x, f^2g^2x) = 0$$

for some $x, y \in X$, then from the nonnegativity of the metric d we see that

$$d(g^2x, g^2f^2y) = 0 = d(f^2x, f^2g^2x)$$

or

$$g^2x = g^2f^2y \quad \text{and} \quad f^2x = f^2g^2x. \tag{6}$$

If $g^2x = f^2x = p$, it follows from the weak** commutativity (3) that $f^2p = g^2p$, that is p is a coincidence point of f^2 and g^2. This together with (6) implies that p is a fixed point of f^2 and hence of g^2. Thus p is a common fixed point of f^2 and g^2. Then again from (3), we obtain that p is a common fixed point of f and g.

Therefore Theorem 1.3 is appropriately modified in the following way:

Theorem 1.4. Let f and g be self-maps on a complete metric space X satisfying the inclusion (4) and the rational inequality (5) for all $x, y \in X$, except $d(g^2x, g^2f^2y) + d(f^2x, f^2g^2x) = 0$ with $f^2x = g^2x$, where $a, b \geq 0$ such that $a + b < 1$. If g is continuous and (f, g) is weak** commuting, then f and g will have a unique common fixed point.

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