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# Characterization of incidence algebras

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**Abstract**

Spiegel and O'Donnell give a characterization of algebras of $n \times n$ matrices which are isomorphic to Incidence Algebras of partially ordered sets with $n$ elements. We generalize this result to get a characterization of algebras of endomorphisms to be isomorphic to Incidence algebras of lower finite partially ordered sets ([3]). In this paper we are giving a characterization of incidence algebras of upper finite partially ordered sets over fields in terms of opposite algebras of subalgebras of endomorphism algebras.

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**Keywords:** incidence algebra, opposite algebra, upper finite.

## 1. Preliminaries

A partially ordered set $X$ is said to be locally finite if, the subset $X_{yz} = \{x \in X : y \leq x \leq z\}$ is finite for each $y \leq z \in X$. $X$ is said to be lower finite if the subset $X_z = \{x \in X : x \leq z\}$ is finite for each $z \in X$ and is said to be upper finite if the subset $X_z = \{x \in X : z \leq x\}$ is finite for each $z \in X$.

**Definition 1.1.** Let $(X, \leq)$ be a partially ordered set. Then the partially ordered set $(X^{op}, \leq^{op})$ such that $x \leq^{op} y$ in $X^{op}$ if and only if $y \leq x$ in $X$ is called the opposite or dual of the given partially ordered set $X$.

**Remark 1.2.**

1. Whenever $X$ is a locally finite partially ordered set, $X^{op}$ will also be locally finite.
2. Whenever $X$ is a lower finite partially ordered set, $X^{op}$ will be upper finite and vice versa.

**Definition 1.3.** The *incidence algebra* $I(X, R)$ of a locally finite partially ordered set $X$ over a commutative ring with identity $R$ is

$$I(X, R) = \{f : X \times X \to R : f(x, y) = 0, if\ x \nleq y\}$$

with operations given by

$$(f + g)(x, y) = f(x, y) + g(x, y)$$
$$(f.g)(x, y) = \sum_{x \leq z \leq y} f(x, z).g(z, y)$$
$$(r.f)(x, y) = r.f(x, y)$$

for $f, g \in I(X, R)$ with $r \in R$ and $x, y, z \in X$.

If $A \subseteq X$ then the function $\delta_A \in I(X, R)$ is defined by

$$\delta_A(x, y) = \{ \begin{matrix} 1, & \text{if } x = y \in A \\ 0, & \text{otherwise} \end{matrix}$$

is called *characteristic function* of $A$. If $A = X$, then we let $\delta_A = \delta$ and this is the identity element of $I(X, R)$.

We define $\delta_{xy} \in I(X, R)$ by

$$\delta_{xy}(u, v) = \{ \begin{matrix} 1, & \text{if } u = x \text{ and } v = y \\ 0, & \text{otherwise} \end{matrix}$$

and write $e_x$ for $\delta_{xx}$. If $X$ is a countable partially ordered set, then the elements of $X$ can be indexed by rational numbers such that $x_i \leq x_j$ implies that $i \leq j$.

The *Jacobson Radical*, denoted by $J(T)$ of a ring with identity is the intersection of all its maximal right ideals. This is always a two sided ideal and it is the largest ideal $J$ of the ring T such that $1 - t$ is invertible for all $t \in J$. It is proved that the Jacobson Radical of an incidence algebra contains all the functions $f \in I(X, R)$ such that $f(x, x) \in J(R)$ for each $x \in X$. So we have,

**Proposition 1.4.** Let $X$ be a locally finite partially ordered set and $R$ a commutative ring with identity. Then $I(X, R)/J(I(X, R)) \cong \prod_{x \in X} R/J(R)$

Already there is a result,

**Proposition 1.5.** Let $X$ be a locally finite partially ordered set and $R$ a commutative ring with identity. Then $I(X, R)$ is isomorphic to a sub ring of $M_{|X|}(R)$.

Then a natural question arises is that, which subalgebras of $M_{|X|}(R)$ are incidence algebras? For incidence algebras of finite posets over a field, we have the following characterization,

**Theorem 1.6.** Let $K$ be a field and $S$ a subalgebra of $M_n(K)$. Then there is a partially ordered set $X$ of order $n$ such that $I(X, K) \cong S\ iff$

(1) $S$ contains n pairwise orthogonal idempotents, and

(2) $\frac{S}{J(S)}$ is commutative.

As mentioned above, an incidence algebra of a partially ordered set with $n$ elements, over a field $F$, can be viewed as a sub algebra of $M_nF$. It is also given that which sub algebras of $M_nF$ will have the structure of an incidence algebra. We have given a necessary and sufficient condition for a sub algebra of an endomorphism algebra to have the structure of incidence algebra of a lower finite partially ordered set over a field $F$.

**Theorem 1.7. [3]** Let $V$ be a $F-$vector space with dimension $|X|$, for a suitable set $X$. Let $S$ be a sub algebra of $End_F V$. Then there exists a lower finite partial ordering in $X$ such that $S \cong I(X, F)$ if and only if,

1. $1 \in S$
2. $S/J(S)$ is commutative
3. For each $x \in X$, there is an $E_x \in S$ of rank 1, such that $E_x.E_y = \delta_{xy}E_x$ and $\bigoplus_{x\in X} E_x(V) = V$
4. $X_y = \{z \in X \mid E_z.S.E_y \neq 0\}$ is finite for each $y \in X$.

**Definition 1.8.** For any $F-$ algebra $A$ we define the opposite algebra $A^{op}$ of $A$ to be the $F-$ algebra whose underlying set and vector space structure are just those of $A$, but the multiplication $*$ in $A^{op}$ is defined by $a * b = ba$.

## 2. Main Result

In this section we are reaching at a characterization of incidence algebras of upper finite partially ordered sets over a field $F$. For that first we prove the following result.

**Theorem 2.1.** Let $X$ be a locally finite partially ordered set and $R$ a commutative ring with identity. Then

$$(I(X, R))^{op} \cong I(X^{op}, R)$$

*Proof.* Consider the mapping $\phi : I(X, R) \to I(X^{op}, R)$ by $\phi(f)(i, j) = f(j, i)$. Clearly $\phi$ will be a bijection and $\phi(f + g) = \phi(f) + \phi(g)$, and $\phi(cf) = c\phi(f)$.

$$(\phi(fg))(i, j) = (fg)(j, i) = \sum_{j\leq k\leq i} f(j, k)g(k, i)$$

$$= \sum_{i\leq^{op}k\leq^{op}j} (\phi(f))(k, j)(\phi(g))(i, k)$$

$$= \sum_{i \leq^{op} k \leq^{op} j} (\phi(g))(i,k)(\phi(f))(k,j)$$

$$= (\phi(g)\phi(f))(i,j)$$

So that $\phi$ is an anti-isomorphism. Now define $\psi : (I(X,R))^{op} \to I(X^{op}, R)$ as $\psi(f) = \phi(f)$. So that $\psi(f * g) = \phi(gf) = \phi(f)\phi(g) = \psi(f).\psi(g)$. Hence $\psi$ is an isomorphism between $(I(X,R))^{op}$ and $I(X^{op}, R)$. ■

Connecting Theorems 2 and 3, and recalling the fact that whenever $X$ is an upper finite partially ordered set $X^{op}$ is a lower finite partially ordered set, we get a characterization of incidence algebras of upper finite partially ordered sets in terms of opposite algebras of sub algebras of endomorphism algebras as follows:

**Theorem 2.2.** Let $V$ be a $F$-vector space with dimension $|X|$, for a suitable set $X$. Let $S$ be a sub algebra of $End_F V$. Then there exists an upper finite partial ordering in $X$ such that $S^{op} \cong I(X,F)$ if and only if,

1. $1 \in S$.
2. $S/J(S)$ is commutative.
3. For each $x \in X$, there is an $E_x \in S$ of rank 1, such that $E_x.E_y = \delta_{xy} E_x$ and $\bigoplus_{x \in X} E_x(V) = V$.
4. $X_y = \{z \in X \mid E_z.S.E_y \neq 0\}$ is finite for each $y \in X$.

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