Fuzzy Semi Continuity and Fuzzy Weak-Continuity

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Abstract

A function \(f\) of a fuzzy topological space \(X\) into a fuzzy topological space \(Y\) to be fuzzy weakly-continuous if for each \(x \in X\) and each fuzzy open neighborhood \(V\) of \(f(x)\) there exists a fuzzy open neighborhood \(U\) of \(x\) such that \(f(U) \subset Fcl(V)\). where \(Fcl(V)\) denotes the fuzzy closure of \(V\).

Definition: (a)
A fuzzy subset \(S\) of a fuzzy topological space \(X\) is said to be fuzzy semi-open if there exists a fuzzy open set \(U\) of \(X\) such that \(U \subset S \subset Fcl(U)\). The family of all fuzzy semi-open sets in \(X\) is denoted by \(FSO(X)\).

Definition: (b)
A function \(f: X \rightarrow Y\) to be fuzzy semi-continuous if \(f^{-1}(V) \in FSO(X)\) for every fuzzy open set \(V\) of \(Y\). It has been known that the fuzzy semi-continuity is equivalent to the fuzzy quasi-continuity.

Definition: (c)
A function \(f: X \rightarrow Y\) to be fuzzy semi-open if \(f(U) \in FSO(Y)\) for every fuzzy open set of \(U\) of \(X\).

Definition: (d)
A function \(f: X \rightarrow Y\) to be fuzzy irresolute (resp. fuzzy pre-semi-open) if for each \(V \in FSO(Y)\) (resp. \(U \in FSO(X)\)), \(f^{-1}(V) \in FSO(X)\) (resp \(f(U) \in FSO(Y)\)).

The purpose of the present paper is to investigate the interrelation among the fuzzy weak-continuity, the fuzzy semi-continuity and some fuzzy weak forms of fuzzy open functions.

A fuzzy semi-continuous function is fuzzy irresolute if it is either fuzzy weakly-open injective or fuzzy almost-open.

A fuzzy semi-open function is fuzzy pre-semi-open if it is either fuzzy weakly-continuous or fuzzy almost-continuous.

A fuzzy semi-continuous function is fuzzy weakly-continuous if the domain is extremely disconnected.
1. FUZZY IRRESOLUTE FUNCTIONS

**Definition 1:1**
A function \( f: X \to Y \) is said to be fuzzy weakly-open if 
\[ f(U) \subseteq \text{Int}(f(F \text{cl}(U))) \]
for every fuzzy open set \( U \) of \( X \).

**Definition 1:2**
A function \( f: X \to Y \) is said to be fuzzy almost-open for every fuzzy regular open set \( U \) of \( X \), \( f(U) \) is fuzzy open in \( V \).

**Definition 1:3**
A function \( f: X \to Y \) is said to be fuzzy almost-open if 
\[ f^{-1}(F \text{cl}(V)) \subseteq F \text{cl}(f^{-1}(V)) \]
for every fuzzy open set \( V \) of \( Y \).

**Lemma 1:4**
If \( f: X \to Y \) is a fuzzy almost-open function then it is fuzzy weakly-open.

**Proof:**
Let \( U \) be a fuzzy open set of \( X \). Since \( f \) is fuzzy almost-open, 
\[ f(\text{Int}(F \text{cl}(U))) \]
is fuzzy open in \( Y \) and hence 
\[ f(U) \subseteq f(\text{Int}(F \text{cl}(U))) \subseteq \text{Int}(f(F \text{cl}(U))). \]
The converse to Lemma 1:4 is not necessarily true.

**Example:**
Let \( X = \{ a, b, c, d \} \) & \( \sigma = \{ X, \{ a, b, d \}, \{ a, b \}, \{ d \}, 0 \} \). 
Let \( Y = \{ x, y, z \} \) & \( \tau = \{ Y, \{ x, y \}, \{ y, z \}, \{ y \}, \{ z \}, 0 \} \). 
Let \( f: (X, \sigma) \to (Y, \tau) \) be a function defined as follows 
\[ f(a) = x \quad f(b) = z, \quad f(c) = f(d) = y. \]
Then \( f \) is fuzzy weakly-open but it is not fuzzy almost-open.

**Definition 1:5**
A function \( f: X \to Y \) is said to be fuzzy somewhat continuous if for each fuzzy open \( V \) of \( Y \) with \( f^{-1}(V) \neq 0 \) there exists a fuzzy open set \( U \) of \( X \) such that 
\[ 0 \neq U \subseteq f^{-1}(V). \]

**Theorem 1:6**
If \( f: X \to Y \) is a fuzzy weakly-open somewhat continuous injection then it is fuzzy irresolute.

**Proof:**
Let \( V \in \text{FSO}(Y) \) and \( x \in f^{-1}(V) \). Put \( y = f(x) \) and let \( U \) be any fuzzy open neighborhood of \( x \), since \( f \) is fuzzy weakly-open, we have 
\[ y \in f(U) \cap V \subseteq \text{Int}(f(F \text{cl}(U))) \cap V \in \text{FSO}(Y). \]
There exists a fuzzy open set \( G \) such that 
\[ 0 \neq G \subseteq \text{Int}(f(F \text{cl}(U))) \cap V. \]
Since \( f \) is fuzzy some what continuous and \( f^{-1}(G) \neq 0 \), there exists an fuzzy open set \( W \) of \( X \) such that \( 0 \neq W \subseteq f^{-1}(G) \). 
Therefore, we obtain \( W \subseteq F \text{cl}(U) \cap f^{-1}(V) \) and hence 
\[ W \subseteq F \text{cl}(U) \cap \text{Int}(f^{-1}(V)) \] because \( f \) is injective. Thus, we have 
\[ 0 \neq F \text{cl}(U) \cap \text{Int}(f^{-1}(V)) \text{ and hence} \]
\[ 0 \neq U \cap \text{Int}(f^{-1}(V)). \]
This shows that 
\[ x \in F \text{cl}(U) \cap \text{Int}(f^{-1}(V)) \text{ and } f^{-1}(V) \in \text{FSO}(X). \]
Theorem 1.7
If a function $f: X \to Y$ is a fuzzy almost open and fuzzy semi-continuous then it is fuzzy irresolute.

Proof:
Let $V \in \text{FSO} (Y)$. Then there exists a fuzzy open set $G$ of $Y$ such that $G \subset V \subset \text{Fcl}(G)$, hence $f^{-1}(G) \subset f^{-1}(V) \subset f^{-1}(\text{Fcl}(G))$. Since $f$ is fuzzy semi-continuous, $f^{-1}(G) \in \text{FSO} (X)$ and hence $f^{-1}(G) \subset \text{Fcl}(\text{Int}(f^{-1}(G)))$.

Now, Put $F= Y-f(X-\text{Fcl}(\text{Int}(f^{-1}(G))))$. Then $F$ is fuzzy closed in $Y$ because $f$ is fuzzy almost open and $\text{Fcl}(\text{Int}(f^{-1}(G)))$ is fuzzy regular closed in $X$. By a straightforward calculation we obtain $G \in F$ and $f^{-1}(F) \subset \text{Fcl}(\text{Int}(f^{-1}(G)))$.

Therefore, we have $f^{-1}(\text{Fcl}(G)) \subset \text{Fcl}(f^{-1}(G))$.

Since $f^{-1}(G) \in \text{FSO} (X)$, we obtain $f^{-1}(V) \in \text{FSO} (X)$.

Lemma 1.8:
If a fuzzy topological space $X$ is extremely disconnected then $\text{Fcl}(U)=U$ for every $U \in \text{FSO} (X)$.

Proof:
In general, we have $S \subset \text{Fcl}(S)$ for every fuzzy subset $S$ of $X$. Thus we shall Show that $U \supset \text{Fcl}(U)$ for each $U \in \text{FSO} (X)$.

Let $0 \neq U \in \text{FSO} (X)$ and $x \not\in U$, then there exists a $V \in \text{FSO} (X)$ such that $x \in V$, & $V \cap U = 0$; hence $\text{Int}(V) \cap \text{Int}(U) = 0$. Since $X$ is extremely disconnected, we have $\text{Fcl}(\text{Int}(V)) \cap \text{Fcl}(\text{Int}(U)) = 0$. Therefore, we have $x \not\in \text{Fcl}(\text{Int}(U)) = \text{Fcl}(U)$.

Theorem 1.9
If a fuzzy topological space $Y$ is extremely disconnected and a function $f: X \to Y$ is fuzzy semi-open fuzzy semi-continuous then $f$ is fuzzy irresolute.

Proof:
Let $V \in \text{FSO} (Y)$. There exists a fuzzy open set $G$ of $Y$ such that $G \subset V \subset \text{Fcl}(G)$, hence $f^{-1}(G) \subset f^{-1}(V) \subset f^{-1}(\text{Fcl}(G))$. Since $Y$ is extremely disconnected, we have $G = \text{Fcl}(G)$ by lemma 1.8, since $f$ is fuzzy semi-open then $f^{-1}(G) \subset \text{Fcl}(f^{-1}(G))$.

Therefore we obtain $f^{-1}(\text{Fcl}(G)) \subset \text{Fcl}(f^{-1}(G))$. Since $f$ is fuzzy semi-continuous, $f^{-1}(G) \in \text{FSO} (X)$ and hence $f^{-1}(V) \in \text{FSO} (X)$.

2. FUZZY PRE-SEMI-OPEN FUNCTIONS
Definition: 2.1
A function $f: X \to Y$ is said to be fuzzy almost-continuous if for each $x \in X$ and each fuzzy neighborhood $V$ of $f(x)$, $\text{Fcl}(f^{-1}(V))$ is a fuzzy neighborhood of $x$.

Definition: 2.2
A function $f: X \to Y$ is said to be some what fuzzy open if for each nonempty fuzzy open set $U$ of $X$, there exists a fuzzy open set $V$ of $Y$ such that $0 \neq V \subset f(U)$.

Theorem 2.3
If a function $f: X \to Y$ is fuzzy weakly-continuous somewhat fuzzy open, then it is fuzzy pre-open.
Proof:
Let \( A \in \text{FSO} (X) \) and \( y \in f(A) \). Let \( V \) be any fuzzy open neighborhood of \( y \). There exists \( x \in A \) such that \( y = f(x) \). Since \( f \) is fuzzy weakly-continuous, there exists a fuzzy open neighborhood \( U \) of \( x \) such that \( f(U) \subseteq \text{Fcl} (V) \).

Since \( x \notin U \cap A \in \text{FSO} (X) \) there exists a fuzzy open set \( W \) of \( X \) such that \( 0 \neq W \subseteq U \cap A \). Moreover, Since \( f \) is fuzzy some what fuzzy open, there exists \( \exists \) fuzzy open set \( G \) of \( Y \) such that \( 0 \neq G \subseteq f(W) \), hence \( G \subseteq \text{Fcl} (V) \cap (f(A)) \). Therefore, we have \( G \subseteq \text{Fcl} (V) \cap (f(A)) \).

This shows that \( y \in \text{Fcl} (\text{Int}(f(A))) \) and hence \( f(A) \subseteq \text{Fcl} (\text{Int}(f(A))) \).

Corollary: 2.4
Every Fuzzy Weakly-continuous fuzzy semi-open function is fuzzy pre-semi-open.

Proof:
Since every fuzzy semi-open function is somewhat fuzzy open, this is an immediate consequence of Theorem 2:3.

Theorem 2:5
If a function \( f: X \to Y \) is fuzzy almost-continuous fuzzy semi open then it is fuzzy pre-semi-open.

Proof:
Let \( U \in \text{FSO} (X) \). There exists a fuzzy open set \( G \) of \( X \) such that \( G \subseteq U \subseteq \text{Fcl} (G) \).

Since \( f \) is fuzzy almost-continuous, we have \( f(\text{Fcl} (G) \subseteq \text{Fcl} (f(G)) \) and hence \( f(G) \subseteq f(U) \subseteq \text{Fcl} (f(G)) \). Since \( f \) is fuzzy semi-open, we obtain \( f(G) \in \text{FSO} (Y) \) and \( f(U) \in \text{FSO} (Y) \).

Theorem 2.6
If a fuzzy topological space \( X \) is extremally disconnected and a function \( f: X \to Y \) is fuzzy semi-continuous fuzzy semi-open, then \( f \) is fuzzy pre-semi-open.

Proof:
Let \( U \in \text{FSO} (X) \). There exists a fuzzy open set \( G \) of \( X \) such that \( G \subseteq U \subseteq \text{Fcl} (G) \).

Since \( X \) is extremally disconnected. We have \( \text{Fcl} (G) = G \) by lemma 1.8 since \( f \) is fuzzy semi-continuous, we obtain \( f(G) \subseteq \text{Fcl} (f(G)) \) and hence \( f(G) \subseteq f(U) \subseteq \text{Fcl} (f(G)) \). Since \( f \) is fuzzy semi-open, we have \( f(G) \in \text{FSO} (Y) \) and \( f(U) \in \text{FSO} (Y) \).

Example: 2:7
Let \( X = Y = \{a, b, c, d\} \sigma = \{ X, \{a,b\}, \{a\}, \{b\}, o \} \) and \( \tau = \{ Y, \{b,c,d\}, \{a,b\} \{a\}, \{b\}, 0\} \)

Let \( f: (X, \sigma) \to (Y, \tau) \) be the identify function. Then \( f \) is fuzzy open and fuzzy semi-continuous but it is not fuzzy pre-semi-open.