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Fixed Point Theorems for Weakly Reciprocally Continuous Mappings

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Abstract

In the present paper we prove common fixed point theorems for recently introduced notion of weakly reciprocally continuous selfmappings, which is independent of the known continuity definitions. As an application of weak reciprocal continuity we prove common fixed point theorems under contractive condition of compatible continuous mappings as well as discontinuous mappings.

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1. Introduction

In recent years the study of common fixed points of contractive type mappings had emerged as an area of intense research activity and a number of interesting fixed point theorems have been reported by various authors. The common fixed point theorems invariably require a commutativity condition, a continuity condition and a contractive condition. A new fixed point theorem can be obtained by weakening one or more of these assumptions. In this paper we take up the continuity condition and weaken the condition in as much as the mappings become discontinuous at the common fixed point. For this purpose we take up recently introduced notion of weakly reciprocally continuous selfmappings, which is independent of the known continuity definitions.

Two selfmaps f and g of a metric space (X, d) are called weakly reciprocally continuous [3], if $\lim_n fgx_n=ft$ or $\lim_n gfx_n=gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n=\lim_n gx_n=t$ for some t in X.

If both f and g are continuous they are reciprocally continuous and reciprocally continuous mappings are obviously weak reciprocally mappings.

The notion of compatible maps was introduced by Jungck [2] in 1986 by generalizing the concept of commutativity or say generalizing the concept of weak commutativity. Two selfmaps f and g of a metric space (X, d) are called compatible if $\lim_n d(fgx_n, gfx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n=t$ for some t in X. Right since the introduction of compatibility, the study of common fixed points is centered on the compatible mappings and it has become an area of vigorous research activity. Albeit, the study of noncompatible maps is equally interesting and various fruitful results have been obtained using the aspect of noncompatibility.

In 1994, Pant [5] has further generalized the notion of weakly commuting maps and introduced the notion of R-weakly commuting mappings. Two selfmappings f and g of a metric space X are called *R-weakly commuting* at a point x in X if $d(fgx, gfx) \leq Rd(fx, gx)$ for some $R> 0$. The maps f and g are called *R- weakly commuting* on X if given x in X there exists $R > 0$ such that

$$d(fgx, gfx) \leq Rd(fx, gx).$$

From the above definition it is obvious that f and g can fail to be pointwise R-weakly commuting only if there exists some x in X such that fx=gx but $fgx \neq gfx$, that is, only if they possess a coincident point at which they do not commute. It may be observed that compatibility implies pointwise R-weak commutativity since compatible maps commute at their coincidence points. The converse, however, is not true.

There are a number of generalizations also the concept of commutativity. It is, however, relevant to mention here that both commutativity and weak commutativity are independent of the notion of weakly reciprocally continuity. We cite few examples to show that if two maps are compatible or weakly compatible they are not necessarily weakly reciprocally continuous and vice versa.

Example 1: Let X=[2, 20] and d be the usual metric on X. Define $f,g : X \rightarrow X$ by

$$fx =6, \text{ if } 2 \leq x < 5, \ fx=2, \text{ if } x \geq 5,$$
$$gx=2, \text{ if } 2 \leq x < 5, \ gx=x-3 \text{ if } x \geq 5,$$

In this example f5=2=g5 but gf5=2, fg5=6. Thus f and g do not commute at their coincidence point x=5. Let us now consider the sequence $\{x_n=5+ 1/n : n > 1\}$, then $\lim_n fx_n= 2$, $\lim_n gx_n=2$, $\lim_n fgx_n=6=f2$ and $\lim_n gfx_n=2=g2$. Thus f and g are weakly *reciprocally continuous* but are neither compatible or weakly compatible.

Example 2: Let X=[2, 20] and d be the usual metric on X. Define $f,g : X \rightarrow X$ by

$$f2 =2, \ fx=2x + 1 \text{ if } 2 < x < 5, \qquad fx=(x - 1)/2 \text{ if } x > 5$$

$$g2=2, \ gx=(x + 8)/2 \text{ if } 2 < x < 5, \quad gx=x-3 \text{ if } x \geq 5$$

Then f and g are compatible but not weakly *reciprocally continuous.* Let us now consider the sequence $\{x_n=5+ 1/n : n > 1\}$, then $\lim_n fx_n= 2$, $\lim_n gx_n=2$, $\lim_n fgx_n=5 \neq f2$ and $\lim_n gfx_n=5 \neq g2$. For the sequence $\{x_n=2 + 1/n : n > 1\}$, then $\lim_n fx_n= 5$, $\lim_n gx_n=5$, $\lim_n fgx_n=2 \neq f5$ and $\lim_n gfx=2 \neq g5$. Thus f and g are compatible but not weakly *reciprocally continuous.*

It is relevant to mention here that if two maps are weakly reciprocally continuous they need not be continuous. In fact the mappings involved in this example are discontinuous at the common fixed point.

In the following pages we prove common fixed point theorems under minimal commutativity conditions using the notion of compatibility and weakly reciprocal continuity. Theorem 2 is slightly improved version of theorem 1 In theorem 3, by using the notion of property (E.A), we further modify our results.

2. Main Results

Theorem 1: Let f and g be R-weakly commuting pair of selfmappings of a complete metric space (X,d) such that

i. $\overline{fX} \subset gX$, where fX denotes the closure of range of f,

ii. $d(fx, gx) \leq kd(gx,gy)$, $k > 0$, and

iii. $d(fx, ffx) < \max\{d(gx,gfx), d(ggx,gfx), d(fx,gx), d(ffx,gfx), d(fx,gfx), d(gx,ffx)$, whenever $fx \neq ffx$.

If f and g are compatible and weakly reciprocally continuous then f and g have a unique common fixed point.

Proof: Let x_0 be any point in X. Then $fX \subset gX$, define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule $s_n=fx_n=gx_{n+1}$, n=0,1,2,….

We claim that $\{ s_n \}$ is a Cauchy sequence. Using (ii) we obtain

$$d(fx_n, fx_{n+1}) \leq \{d(gx_n, gx_{n+1}\},$$

$$d(s_n, s_{n+1}) \leq \{d(s_{n-1}, s_n\},$$

$$d(s_n, s_{n+1}) \leq d(s_{n-1}, s_n) \qquad \text{……..(1)}$$

To prove that $\{ s_n \}$ is a Cauchy sequence we prove that (1) is true for all $n \geq n_0$ and for every $m \in N$,

$$d(s_n, s_{n+m}) > 1-\lambda \qquad \text{.......(2)}$$

Here we use induction method

$d(s_n, s_{n+1}) \leq d(s_{n-1}, s_n) \leq d(s_{n-2}, s_{n-1}) \leq \ldots\ldots \leq d(s_0, s_1) \to 1$as $n\to \infty$

i.e. for $0 \leq \lambda <1$, we can choose $n_0 \in N$, such that

$d(s_n, s_{n+1}) < 1-\lambda$

Thus (2) is true for m=1. Suppose (2) is true for m then we shall show that it is also true for m + 1. We have

$$d(s_n, s_{n+m+1}) \leq \{d(s_n, s_{n+m}), d(s_{n+m}, s_{n+m+1})\} < 1-\lambda.$$

Hence (2) is true for m + 1. Thus { s_n } is a Cauchy sequence. By completeness of (X, d), { s_n } converges to some point t in X. Moreover, $\lim_n fx_n = \lim_n gx_n = t$.

Since weakly reciprocally continuous selfmappings of f and g implies that $\lim_n fgx_n = ft$, $\lim_n gfx_n = gt$ for some t in X. Since f and g are compatible, then $\lim_n d(fgx_n, gfx_n)=0$, that is fu=gu. since $fX \subset gX$ there exists a point w in X such that fu=gw. Using (ii) we get

d(fu, gw) ≤ kd(gu,gw)=kd(fu, gw) that is, fu=gw. Thus fu=gu= fw=gw. Pointwise R–weak commutativity of f and g implies that there exists R > 0 such that d(fgu, gfu) ≤ Rd(fu,gu)=0, that is, fgu=gfu and ffu=fgu=gfu=ggu. If fu ≠ ffu=fgu =gfu=ggu. Using (iii), we get

$$d(fu, ffu)=d(gw, ffu) < \max\{d(gfu,gw), d(gfu,gw), d(ffu,gfu), d(fw,gw), d(ffu,gw), d(gfu,fw)=d(fu, ffu)$$

a contradiction. Hence, fu=ffu and fu=ffu=fgu =gfu=ggu. Hence fu is a common fixed point of f and g. The case when fX is a complete subspace of X is similar to the above case since $fX \subset gX$. Hence the theorem.

In the next theorem we replace the condition (iii) of the above theorem.

Theorem 2: Let f and g be R-weakly commuting pair of selfmappings of a complete metric space (X,d) such that

i. $\overline{fX} \subset gX$, where fX denotes the closure of range of f,

ii. $d(fx, gx) \leq kd(gx,gy)$, $k > 0$, and

iii. $d(fx, ffx) > \max\{d(gx,gfx), d(ggx,gfx), d(fx,gx),$

$d(ffx,gfx), d(fx,gfx), d(gx,ffx)$, whenever $fx \neq ffx$. If f and g are compatible and weakly reciprocally continuous then f and g have a unique common fixed point.

The theorem can be proved in similar manner as in Theorem 1. For this we give an example.

Example 3: Let X=[2, 20] and d be the usual metric on X. Define f, g : X → X by fx=2, fx=2 or >2, fx=6 if $2 < x \leq 5$, gx=2, gx=12 if $2 < x \leq 5$, gx=x - 4 if x > 5

Then f and g satisfy all the conditions of above theorem and have a unique common fixed point x=2. In this example f and g are weakly *reciprocally continuous.* To see this let us now consider the sequence $\{x_n\}$ be a sequence in X such that $\lim_n fx_n \to 2$, $\lim_n gx_n \to 2$ for t. Then t=2 and either $x_n=2$ for each n or $x_n=5+ 1/n$ as n > 1, $\lim_n fgx_n \to 2=f2$, $\lim_n gfx_n \to 2=g2$. If $x_n=5+ 1/n$ as n > 1, then $\lim_n fx_n= 2$, $\lim_n gx_n=2 + 1/3n \to 2$, $\lim_n fgx_n=f(2 + 1/3n)=6 \neq f2$ and $\lim_n gfx_n=2=g2$. Thus

$\lim_n gfx_n = g2$ but $\lim_n fgx_n \neq 2$. Hence mappings f and g are weakly reciprocally continuous but not reciprocally continuous.

In the next theorem, using the notion of weak reciprocal continuity, we get the following result. The result can be seen as an answer to the problem of Rhoades [6] regarding existence of contractive definition which ensures the existence of common fixed point but does not force the maps to become continuous at the fixed point.

Theorem 3: Let f and g be R-weakly commuting pair of selfmappings of a complete metric space (X, d) such that

i. $\overline{fX} \subset gX$, where fX denotes the closure of range of f,

ii. $d(fx, fy) \leq kd(gx,gy)$, $k > 0$, and

iii. $d(fx, ffx) < \max\{d(gx,ggx)\}$, whenever $gx \neq ggx$.

If f and g satisfy the property (E.A) and weakly reciprocally continuous then f and g have a unique common fixed point.

Proof: Since f and g satisfy the property (E.A), there exists a sequence $\{x_n\}$ such that $\lim_n fx_n \to t$ and $\lim_n gx_n \to t$ for some t in X. Since $fX \subset gX$, for each $\{x_n\}$ there exists $\{y_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n$. Thus $\lim_n fx_n \to t$ and $\lim_n gx_n \to t$. By (ii) we now get $d(fx_n, fy_n) \leq kd(gx_n, gy_n)$. We get $\lim_n fx_n \to t$, $\lim_n gx_n \to t$ and

$\lim_n fy_n \to t$, $\lim_n gy_n \to t$.

Since weakly reciprocally continuous selfmappings of f and g implies that $\lim_n fgx_n = ft$, $\lim_n gfx_n = gt$ for some t in X. Similarly $\lim_n fgy_n = ft$, $\lim_n gfy_n = gt$.

Since R–weak commutativity of f and g implies that there exists $R > 0$ such that $d(fgy_n, gfy_n) \leq Rd(fy_n, gfx_n)$ on letting $n \to \infty$, we get $fgy_n \to gt$. Using (ii), we get $d(fgy_n, ft) \leq kd(gfy_n, gt)$. On letting $n \to \infty$, we get $d(gt, ft) \leq kd(gt, gt)$. This implies that ft=gt, since $k > 0$. Again, since R–weak commutativity of f and g implies that there exists $R > 0$ such that $d(fgt, gft) \leq Rd(ft,gt)=0$, that is, fgt=gft and fft=fgt=gft=ggt. If ft $\neq$ fft=fgt =gft=ggt. Using (iii), we get

$$d(ft, fft) < \max\{d(gt,ggt)=d(ft, fft),$$

a contradiction. Hence, ft=fft and ft=fft=fgt =gft=ggt. Hence ft is a common fixed point of f and g.

Next suppose $\lim_n fgy_n = ft$, since $fX \subset gX$ there exists a point u in X such that ft=gu. Again we have $fgy_n = ffy_n \to ft$. Thus $fgy_n = ft = gu$ and $fx_n \to gu$.

R–weak commutativity of f and g implies that there exists $R > 0$ such that $d(fgx_n, gfx_n) \leq Rd(fx_n,gx_n)$, on letting $n \to \infty$, we get $gfx_n = gu$ i.e. $ggy_n \to gu$. Using (ii), we get $d(fgy_n, fu) \leq kd(ggy_n, gu)$. On letting $n \to \infty$, we get $d(gu, fu) \leq kd(fu, gu)$, this implies that fu=gu. Again by virtue of R–weak commutativity $d(fgu, gfu) \leq Rd(fu,gu)=0$, that is, fgu=gfu and ffu=fgu=gfu=ggu. If fu $\neq$ ffu=fgu =gfu=ggu. If fu $\neq$ ffu=fgu =gfu=ggu then by using (iii), we get

$$d(fu, ffu) < \max\{d(gu,ggu)\} = d(fu, ffu)$$

a contradiction. Hence, fu=ffu and fu=ffu=fgu =gfu=ggu. Hence fu is a common fixed point of f and g. Hence the theorem.

Example 4: Let X=[2, 20] and d be the usual metric on X. Define f, g : X → X by

f2=2, if x=2 or > 5, fx=6 if 2 < x ≤ 5,
g2=2, gx=x + 4 if 2 < x ≤ 5, gx=(4x + 10) /15 if x > 5.

Then f and g satisfy all the conditions of Theorem 3 and have a unique common fixed point x=2. In this example fX={2} U {6} and gX=[2, 6] U {7}. It may be seen that fX ⊂ gX. It can be verified that f and g are satisfy the property (E.A).

Remark: Aamri and Moutwakil [1] introduced Property (E.A) is more general then the notion of noncompatibility. It is however, worth to mention here that if we take noncompatibility aspect instead of the property (E.A) we can show, in addition, that the mappings are discontinuous at the common fixed point.

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