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Rough Anti-homomorphism on a Rough Group

Neelima C.A.[1]
Department of Mathematics and Statistics,
SNM College Maliankara,
Kerala, India.
E-mail: neelimaasokan@gmail.com

Paul Isaac
Department of Mathematics,
Bharata Mata College Thrikkakara,
Kochi-682 021, Kerala, India.
E-mail: pibmct@gmail.com

Abstract

In this paper, we shall study the concept of rough group, introduce the concept of rough anti-homomorphism and give some properties of rough group with respect to rough anti-homomorphism.

AMS subject classification:
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1. introduction

The theory of Rough set was proposed by Z Pawlak in 1982 [11]. It is emerging as a powerful mathematical tool for imperfect data analysis.The rough set theory is an extension of set theory in which a subset of universe is approximated by a pair of ordinary sets, called upper and lower approximations. A key concept in Pawlak rough set model is an equivalence relation, which are the building blocks for the upper and lower approximations. Combining the theory of rough set with abstract algebra is one of the trends in the theory of rough set. Some authors substituted an algebraic structure for the universal set and studied the roughness in algebraic structure. On the other hand, some authors studied the concept of rough algebraic structures. The concepts of

[1]Corresponding author.

rough group, rough semigroup and rough quotient group are studied in [10], [3] and [5]. B. Davvaz studied roughness in rings [4]. Su-Qing Han [12] proposed the concept of rough ring in an approximation space. He also proposed the concept of rough cosets and rough normal groups. Some properties of rough subrings and rough ideals are studied respectively in [8] and [7]. In this paper, we shall review the concept of rough group, introduce the concept of rough anti-homomorphism and give some properties of them.

In section 2 we give the basic concepts of rough sets. Section 3 deals with the concepts of a rough group and rough homomorphism. In section 4 we define rough anti-homomorphism of rough groups and prove some related results.

2. Basic concepts of Rough Sets

In this section we give the basic concepts of rough sets. For crisp algebraic concepts one may refer the books by Gallian [6] or Artin [1].

Definition 2.1. A pair (U, θ) where $U \neq \phi$ and θ an equivalence relation on U, is called an approximation space.

Definition 2.2. For an approximation space (U, θ) and a subset X of U, the sets

1. $\theta^{-}(X) = \{x \in U : [x]_\theta \cap X \neq \phi\}$
2. $\theta_{-}(X) = \{x \in U : [x]_\theta \subseteq X\}$
3. $BN(X) = \theta^{-}(X) - \theta_{-}(X)$

are called upper approximation, lower approximation and boundary region of X in (U, θ), respectively.

Proposition 2.3. Let $X, Y \subset U$. The following properties hold.

1. $\theta_{-}(X) \subset X \subset \theta^{-}(X)$
2. $\theta_{-}(\phi) = \theta^{-}(\phi) = \phi, \theta_{-}(U) = \theta^{-}(U) = U$
3. $\theta_{-}(X \cap Y) = \theta_{-}(X) \cap \theta_{-}(Y)$
4. $\theta^{-}(X \cap Y) \subset \theta^{-}(X) \cap \theta^{-}(Y)$
5. $\theta_{-}(X) \cup \theta_{-}(Y) \subset \theta_{-}(X \cup Y)$
6. $\theta^{-}(X \cup Y) = \theta^{-}(X) \cup \theta^{-}(Y)$
7. $X \subset Y$ iff $\theta^{-}(X) \subset \theta^{-}(Y)$ and $\theta_{-}(X) \subset \theta_{-}(Y)$

3. Rough Group

Definition 3.1. [10] Let (U, θ) be an approximation space and let $*$ be a binary operations on U. A subset G of U is called a rough group if it satisfies the following properties.

1. $\forall\, x, y \in G, x * y \in \theta^{-}(G)$.
2. Associativity holds in $\theta^{-}(G)$.
3. $\forall\, x \in G, \exists\, e \in \theta^{-}(G)$ such that $x * e = x = e * x$, e is called the rough identity.
4. $\forall\, x \in G, \exists\, y \in G$ such that $x * y = e = y * x$, y is called the rough inverse.

Definition 3.2. [10] A non-empty subset H of a rough group G is called its rough subgroup if it is a rough group itself with respect to the same operations on G.

Definition 3.3. [10] A rough subgroup N of a rough group G is called a rough invariant subgroup if
$\forall x \in G, a * N = N * a$.

Theorem 3.4. [10] A necessary and sufficient condition for a rough subgroup N of G to be a rough invariant subgroup is that $\forall\, a \in G$ and $n \in N, a * n * a^{-1} \in N$.

Definition 3.5. [10] Let (U_1, θ_1) and (U_2, θ_2) be two approximation spaces and $*; *'$ be binary operations over U_1 and U_2 respectively. Let $G_1 \subseteq U_1$ and $G_2 \subseteq U_2$ be two rough groups.

A mapping $\phi : \theta_1^{-}(G_1) \to \theta_2^{-}(G_2)$ satisfying

$$\phi(x * y) = \phi(x) *' \phi(y)$$

$\forall x, y \in \theta_1^{-}(G_1)$, is called a rough homomorphism from G_1 to G_2.

Definition 3.6. A rough homomorphism ϕ from a rough group G_1 to a rough group G_2 is called

- a rough epimorphism (or surjective) if ϕ: $\theta_1^{-}(G_1) \to \theta_2^{-}(G_2)$ is onto. That is $\forall y \in \theta_2^{-}(G_2), \exists\, x \in \theta_1^{-}(G_1)$ such that $\phi(x) = y$
- a rough embedding (or monomorphism) if ϕ: $\theta_1^{-}(G_1) \to \theta_2^{-}(G_2)$ is one-one.
- a rough isomorphism if ϕ: $\theta_1^{-}(G_1) \to \theta_2^{-}(G_2)$ is both one-one and onto.

Definition 3.7. A rough homomorphism is called

- a rough endomorphism if it is a rough homomorphism from the rough group into itself.

- a rough automorphism if it is a rough isomorphism from a rough group into itself.

Theorem 3.8. Let G be a rough group and ϕ_1 and ϕ_2 be two rough homomorphisms on G. Then the composition $\phi_1 o \phi_2$ is a rough homomorphism on G.

Proof. Let G be a rough group and let ϕ_1, ϕ_2 be two rough homomorphisms on G.
Then $\phi_1, \phi_2 : \theta^-(G) \to \theta^-(G)$ such that $\forall\, x, y \in \theta^-(G)$

$$\phi_1(x * y) = \phi_1(x) * \phi_1(y) \quad \text{and} \quad \phi_2(x * y) = \phi_2(x) * \phi_2(y)$$

Now $\forall\, x, y \in \theta^-(G)$

$$(\phi_1 o \phi_2)(x * y) = \phi_1(\phi_2(x * y)) = \phi_1(\phi_2(x) * \phi_2(y)) = (\phi_1 o \phi_2)(x) * (\phi_1 o \phi_2)(y)$$

Therefore, $\phi_1 o \phi_2$ is a rough homomorphism. ■

Theorem 3.9. The rough endomorphisms of a rough group G form a monoid.

Proof. Let G be a rough group and let ϕ_1, ϕ_2 be two rough homomorphisms on G. Then $\phi_1, \phi_2 : \theta^-(G) \to \theta^-(G)$ such that $\forall x, y \in \theta^-(G)$

$$\phi_1(x * y) = \phi_1(x) * \phi_1(y) \quad \text{and} \quad \phi_2(x * y) = \phi_2(x) * \phi_2(y)$$

Now $\forall\, x, y \in \theta^-(G)$

$$(\phi_2 o \phi_1)(x * y) = \phi_2(\phi_1(x * y)) = (\phi_2 o \phi_1)(x) * (\phi_2 o \phi_1)(y)$$

Therefore, $\phi_2 o \phi_1$ is a rough endomorphism on G. Associative property also holds. The identity function I on $\theta^-(G)$ such that $I(x) = x$, is also a rough endomorphism. Hence the theorem is proved. ■

Theorem 3.10. The rough automorphisms of a rough group G form a group.

Proof. Let G be a rough group and let ϕ_1, ϕ_2 be two rough homomorphisms on G.
Then $\phi_1, \phi_2 : \theta^-(G) \to \theta^-(G)$ such that $\forall x, y \in \theta^-(G)$

$$\phi_1(x * y) = \phi_1(x) * \phi_1(y) \quad \text{and} \quad \phi_2(x * y) = \phi_2(x) * \phi_2(y)$$

and ϕ_1, ϕ_2 are one- one and onto.

By theorem 3.8, $\phi_1 o \phi_2$ is a rough homomorphism on G. Since ϕ_2 and ϕ_1 are one-one and onto, $\phi_1 o \phi_2$ is also one-one and onto.Therefore $\phi_1 o \phi_2$ is a rough automorphism on G.

Associative property also holds.The identity function I on $\theta^-(G)$ such that $I(x) = x$, is also a rough automorphism. For any rough automorphism ϕ_1 on G , $\exists$ a rough automorphism ϕ_2 on G such that $\phi_1 o \phi_2 = I$. This completes the proof. ■

4. Anti-homomorphism of Rough Groups

Let (U_1, θ_1) and (U_2, θ_2) be two approximation spaces and $*; *'$ be binary operations over U_1 and U_2 respectively. Let $G_1 \subseteq U_1$ and $G_2 \subseteq U_2$ be two rough groups.

Definition 4.1. A mapping $\phi : \theta_1^-(G_1) \to \theta_2^-(G_2)$ satisfying

$$\phi(x * y) = \phi(y) *' \phi(x)$$

$\forall x, y \in \theta_1^-(G_1)$, is called a rough anti-homomorphism from G_1 to G_2.

Theorem 4.2. Let G_1 and G_2 be two rough groups and ϕ be a rough anti-epimorphism from G_1 to G_2. If $*$ satisfies the commutative law, then $*'$ also satisfies the commutative law.

Proof. For $x', y' \in \theta_2^-(G_2)$, $\exists x, y \in \theta_1^-(G_1)$ such that $\phi(x) = x'$ and $\phi(y) = y'$. Since $x * y = y * x$ we have

$$x' *' y' = \phi(x) *' \phi(y) = \phi(y * x) = \phi(x * y) = \phi(y) *' \phi(x) = y' *' x'$$

Therefore $*'$ is commutative. ■

Theorem 4.3. Let ϕ be a rough anti-homomorphism from G_1 to G_2. Then

1. If e is the rough identity of G_1, then $\phi(e)$ is the rough identity of $\phi(G_1)$.
2. If $x \in G_1$, then $\phi(x^{-1}) = (\phi(x))^{-1}$.

provided $\phi(\theta_1^-(G_1)) = \theta_2^-(\phi(G_1))$.

Proof. For $x' \in \phi(G_1)$, $\exists x \in G_1$ such that $\phi(x) = x'$

1. Because R_1 is a rough group, $e \in \theta_1^-(G_1)$ such that $x * e = x = e * x$. Therefore

   $$x' *' \phi(e) = \phi(x) *' \phi(e) = \phi(e * x) = \phi(x * e) = \phi(e) *' \phi(x) = \phi(e) *' x'$$

   Also $\phi(e) \in \phi(\theta_1^-(G_1))$. Because $\phi(\theta_1^-(G_1)) = \theta_2^-(\phi(G_1))$, $\phi(e) \in \theta_2^-(\phi(G_1))$. Therefore $\phi(e)$ is the rough identity of $\phi(G_1)$.

2. Because G_1 is a rough group, $x^{-1} \in G_1$ such that $x * x^{-1} = e = x^{-1} * x$. So we get

   $$\phi(x) *' \phi(x^{-1}) = \phi(x^{-1} * x) = \phi(x * x^{-1}) = \phi(x^{-1}) *' \phi(x)$$

   Thus we get by definition $\phi(x^{-1}) = (\phi(x))^{-1}$. ■

Theorem 4.4. Let ϕ be a rough anti- homomorphism from G_1 to G_2 . Then $\phi(G_1)$ is a rough group if $\phi(\theta_1^-(G_1)) = \theta_2^-(\phi(G_1))$.

Proof. For $x', y' \in \phi(G_1)$, $\exists\ x, y \in G_1$ such that $\phi(x) = x'$ and $\phi(y) = y'$.

1. Now
$$\phi(y * x) = \phi(x) *' \phi(y) = x' *' y'$$
since $\phi(y * x) \in \phi(\theta_1^-(G_1))$, we have $\phi(y * x) \in \theta_2^-(\phi(G_1))$. That is $x' *' y' \in \theta_2^-(\phi(G_1))$

2. Since $e \in \theta_1^-(G_1)$, we have $\phi(e) \in \phi(\theta_1^-(G_1)) = \theta_2^-(\phi(G_1))$. Therefore $\phi(x) *' \phi(e) = \phi(e * x) = \phi(x * e) = \phi(e) *' \phi(x)\ \forall\ \phi(x) \in \phi(G_1)$.

3. Since G_1 is a rough group, $\forall\ x, y, z \in G_1$, $x * (y * z) = (x * y) * z$
$$\phi(x * (y * z)) = \phi(y * z) *' \phi(x) = (\phi(z) *' \phi(y)) *' \phi(x)$$
$$\phi((x * y) * z) = \phi(z) *' \phi(x * y) = \phi(z) *' (\phi(y) *' \phi(x))$$
Therefore $\phi(x) *' (\phi(y) *' \phi(z)) = (\phi(x) *' \phi(y)) *' \phi(z)$

4. Because G_1 is a rough group, for $x \in G_1$, $\exists\ x^{-1} \in G_1$ such that $x * x^{-1} = e = x^{-1} * x$. Thus $\phi(x)^{-1} \in \phi(G_1)$ such that
$$\phi(x) *' \phi(x^{-1}) = \phi(x^{-1} * x) = \phi(x * x^{-1}) = \phi(x^{-1}) *' \phi(x)$$
So by definition $\phi(x^{-1}) = (\phi(x))^{-1}$.

Therefore, $\phi(G_1)$ is a rough group. ■

Theorem 4.5. Let H be a rough subgroup of G_1 and let ϕ be a rough anti-homomorphism from G_1 to G_2. Then $\phi(H)$ is a rough subgroup of G_2 if $\phi(\theta_1^-(H)) = \theta_2^-(\phi(H))$.

Proof. The proof is similar to that of theorem 4.4. ■

Theorem 4.6. Let H_2 be a rough subgroup of G_2 and let ϕ be a rough anti-homomorphism from G_1 to G_2 . Then $H_1 = \phi^{-1}(H_2)$ is a rough subgroup of G_1 if $\phi(\theta_1^-(H_1)) = \theta_2^-(\phi(H_1))$.

Proof. Since $H_1 = \phi^{-1}(H_2)$, we have $\phi(H_1) = H_2$, and so $\theta_2^-(H_2) = \theta_2^-(\phi(H_1)) = \phi(\theta_1^-(H_1))$

1. $\forall x, y \in H_1$, we have $\phi(x), \phi(y) \in H_2$. Since H_2 is rough group, we get $\phi(y) *' \phi(x) \in \theta_2^-(H_2)$. That is, $\phi(x * y) \in \phi(\theta_1^-(H_1))$. Thus we get $x * y \in \theta_1^-(H_1)$.

2. $\forall x \in H_1$, we have $\phi(x) \in H_2$. Since H_2 is a rough group, $\phi(x^{-1}) = (\phi(x))^{-1} \in H_2$. That is $\phi(x^{-1}) \in \phi(H_1)$. Thus $x^{-1} \in H_1$

Therefore H_1 is a rough subgroup of G_1. ■

Theorem 4.7. Let N be a rough invariant subgroup of G_1 and let ϕ be a rough anti-homomorphism from G_1 to G_2. Then $\phi(N)$ is a rough invariant subgroup of G_2 if $\phi(\theta_1^-(N)) = \theta_2^-(\phi(N))$ and $\phi(G_1) = G_2$.

Proof. The proof is similar to that of theorem 4.4.

Theorem 4.8. Let N_2 be a rough invariant subgroup of G_2. Then $N_1 = \phi^{-1}(N_2)$ is a rough invariant subgroup of G_1 if $\phi(\theta_1^-(N_1)) = \theta_2^-(\phi(N_1))$.

Proof. The proof is similar to that of theorem 4.6. ∎

Definition 4.9. Let $G_1 \subseteq U_1$ and $G_2 \subseteq U_2$ be two rough groups and ϕ be a rough anti-homomorphism from G_1 to G_2. Then $\{x/\phi(x) = e_2, x \in G_1\}$ where e_2 is the rough identity of G_2, is called rough anti-homomorphism kernel, denoted by $ker\phi$.

Theorem 4.10. Let ϕ be a rough anti-homomorphism from G_1 to G_2. Then rough anti-homomorphism kernel is a rough invariant subgroup of G_1.

Proof. $\forall\, x, y \in ker\phi, \phi(x) = e_2, \phi(y) = e_2$.

1. We have $\phi(x * y) = \phi(y) *' \phi(x) = e_2$. Therefore, $x * y \in ker\phi$
2. Since $\phi(x^{-1}) = (\phi(x))^{-1} = (e_2)^{-1} = e_2$, we get $x^{-1} \in ker\phi$. Therefore, $ker\phi$ is a rough subgroup of G_1.
3. $\forall x \in G$ and $r \in ker\phi$, We have $\phi(x * r * x^{-1}) = \phi(x^{-1}) *' \phi(r) *' \phi(x) = e_2$. Therefore, $x * r * x^{-1} \in ker\phi$.

Therefore by theorem 3.4, $ker\phi$ is a rough invariant subgroup of G_1. ∎

Theorem 4.11. Let ϕ be a rough anti-epimorphism from G_1 to G_2. Then ϕ is a rough anti-isomorphism if and only if rough anti-homomorphism kernel is $\{e\}$, where e denotes the rough identity of G_1.

Proof. It is straight forward. ∎

Theorem 4.12. Let G be a rough group and ϕ_1 and ϕ_2 be two rough anti-homomorphisms on G. Then the composition $\phi_1 o \phi_2$ is a rough homomorphism on G.

Proof. Let G be a rough group and let ϕ_1, ϕ_2 be two rough anti-homomorphisms on G. Then

$$\phi_1, \phi_2 : \theta^-(G) \to \theta^-(G)$$

such that $\forall x, y \in \theta^-(G)$

$$\phi_1(x * y) = \phi_1(y) * \phi_1(x) \quad \text{and} \quad \phi_2(x * y) = \phi_2(y) * \phi_2(x)$$

Now $\forall x, y \in \theta^-(G)$

$$(\phi_1 o \phi_2)(x * y) = \phi_1(\phi_2(x * y)) = \phi_1(\phi_2(y) * \phi_2(x)) = (\phi_1 o \phi_2)(x) * (\phi_1 o \phi_2)(y)$$

Therefore, $\phi_1 o \phi_2$ is a rough homomorphism on G. ∎

Theorem 4.13. Let G be a rough group and ϕ_1 be a rough anti-homomorphism and ϕ_2 be a rough homomorphism on G. Then the composition $\phi_1 o \phi_2$ is a rough anti-homomorphism on G.

Proof. Let G be a rough group and let ϕ_1 be a rough anti-homomorphism on G and ϕ_2 be a rough homomorphism on G.

Then

$$\phi_1, \phi_2 : \theta^-(G) \to \theta^-(G)$$

such that $\forall x, y \in \theta^-(G)$

$$\phi_1(x * y) = \phi_1(y) * \phi_1(x) \quad \text{and} \quad \phi_2(x * y) = \phi_2(x) * \phi_2(y)$$

Now $\forall\ x, y \in \theta^-(G)$

$$(\phi_1 o \phi_2)(x * y) = \phi_1(\phi_2(x * y)) = \phi_1(\phi_2(x) * \phi_2(y)) = (\phi_1 o \phi_2)(y) * (\phi_1 o \phi_2)(x)$$

Therefore, $\phi_1 o \phi_2$ is a rough anti-homomorphism on G. ■

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