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On Contra $\#rg$-Continuous Functions in Topological Spaces

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Abstract

The purpose of this paper is to introduce the notion of contra $\#rg$-continuous functions and to obtain fundamental properties of contra $\#rg$-continuous functions. Also, we discuss the relationships between contra $\#rg$-continuity and other related functions.

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1. Introduction

Dontchev [2] introduced the notions of contra-continuity and strong S-closedness in topological spaces. He defined a function $f : X \to Y$ to be contra-continuous if the preimage of every open set of Y is closed in X. In [2], he obtained very intersting and important results concerning contra-continuity, compactness, S-closedness and strong S-closedness. Recently a new weaker form of this class of functions called contra-semicontinuous functions is introduced and investigated by Dontchev and Noiri [3]. They also introduced the notion of RC-continuity [3] between topological spaces which is weaker than contra-continuity and stronger than B-continuity [13]. In 1999, Jafari [4] introduced and investigated a new class of functions called contra-super-continuous functions which lies between classes of RC-continuous functions and contra-continuous functions. The concepts of $\#rg$-closed sets is introduced and its properties are studied by S. Syed et al. [10] in 2011 for general topological spaces. In this paper we present a new generalization of contra continuity called contra $\#rg$-continuity which is weaker than contra-continuous function. We define this class of functions by the requirement that the inverse image of each open set in the codomain is $\#rg$-closed in the domain.

In the present paper, spaces X and Y always mean topological spaces on which no separation axioms are assumed unless explicity stated. For a subset A of a space X, $cl(A)$ and $int(A)$ represent the *closure of* A and the *interior of* A, respectively. A subset A of a space X is said to be *regular open* (respectively *regular closed*) if $A = int(cl(A))$ (respectively $A = cl(int(A))$) [8]. A subset A is said to be *semi-open* [5] if $A \subset cl(int(A))$. A subset A of a space $(X,\ T)$ is called *regular semi-open* [1] if there is a regular open set U such that $U \subset A \subset cl(U)$. The family of all regular semi-open sets of X is denoted by $RSO(X)$. A subset A of a space X is said to be *regular weakly closed* (briefly, rw*-closed*) [14] (resp. $\#rg$*-closed* [10]) if $cl(A) \subset U$ whenever $A \subset U$ and U is regular semi-open (resp. rw*-open*) in X. The complement of a rw*-closed* (resp. $\#rg$*-closed*) A is rw*-open* (resp. $\#rg$*-open*). The family of all $\#rg$-open (resp. $\#rg$-closed, closed) sets of X containing a point $x \in X$ is denoted by $\#RGO(X, x)$ (resp. $\#RGC(X, x)$, $C(X, x)$). The family of all $\#rg$-open (resp. $\#rg$-closed, closed, semi-open) sets of X is denoted by $\#RGO(X)$ (resp. $\#RGC(X)$, $C(X)$, $SO(X)$). Let X be a topological space and $S \subset X$. Then (i) the set $\cap\{A : S \subset A$ and $A \in \#RGC(X)\}$ is called $\#rg$-closure of S and is denoted by $\#rg\text{-}cl(S)$, (ii) The set $\cup\{A : A \subset S$ and $A \in \#RGO(X)\}$ is called $\#rg$-interior of S and is denoted by $\#rg\text{-}int(S)$ [10].

A subset A of X is $\#rg$-closed in a space X if $A = \#rg\text{-}cl(A)$ [10]. Let A be a subset of a space $(X,\ T)$. The set $\cap\{U : U \in T$ and $A \subset U\}$ is called *kernal of* A and is denoted by $ker(A)$ [6].

A space X is said to be strongly S-closed [2] if every closed cover of X has a finite subcover. A subset A of a space X is said to be strongly S-closed if the subspace A is strongly S-closed. A space X is said to be strongly countably S-closed if every countable cover of X by closed sets has a finite subcover. A space X is said to be strongly S-lindelof if every closed cover of X has a countable subcover. A topological space X is said to be ultra normal [7] if each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets. a topological space X is said to be ultra Hausdorff [7] if for each pair of distinct points x and y in X there exist clopen sets A and B containing x and y, respectively such that $A \cap B = \phi$.

Definition 1.1. A function $f : X \to Y$ is said to be

(1) RC-continuous [3] if $f^{-1}(V)$ is regular closed in X for each open set V of Y,

(2) #rg-continuous [12] if $f^{-1}(V)$ is #rg-open in X for each open set V of Y.

(3) #rg-open [12] if image of each #rg-open set is #rg-open.

Definition 1.2. [10] A space $(X,\ T)$ is called #rg-$T_{\frac{1}{2}}$ if every #rg-closed set is closed.

2. Contra #rg-continuous functions and its relationships with other functions

Definition 2.1. A function $f : X \to Y$ is called **contra #rg-continuous** if $f^{-1}(V)$ is #rg-closed set in X for every open set V of Y.

Theorem 2.2. The following are equivalent for a function $f : X \to Y$:

(1) f is contra #rg-continuous,

(2) the inverse image of every closed set of Y is #rg-open.

Proof. Let U be any closed set of Y. Since $Y \backslash U$ is open, then by (1) it follows that $f^{-1}(Y \backslash U) = X \backslash f^{-1}(U)$ is #rg-closed. This shows that, $f^{-1}(U)$ is #rg-open in X. Converse is similar. ■

Theorem 2.3. Suppose that $\#RGC(X)$ is closed under arbitrary intersections. Then the following are equivalent for a function $f : X \to Y$:

(1) f is contra #rg-continuous,

(2) the inverse image of every closed set of Y is #rg-open,

(3) for each $x \in X$ and each closed set B in Y with $f(x) \in B$, there exists a #rg-open set A in X such that $x \in A$ and $f(A) \subset B$,

(4) $f(\#rg\text{-}cl(A)) \subset ker(f(A))$ for every subset A of X,

(5) $\#rg\text{-}cl(f^{-1}(B)) \subset f^{-1}(ker(B))$ for every subset B of Y.

Proof. (1) $\Rightarrow$ (3) Let $x \in X$ and B be a closed set in Y with $f(x) \in B$. By (1), it follows that $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$ is $\#rg$-closed and so $f^{-1}(B)$ is $\#rg$-open. Take $A = f^{-1}(B)$, we obtain $x \in A$ and $f(A) \subset B$.
(3) $\Rightarrow$ (2) Let B be a closed set in Y with $x \in f^{-1}(B)$. Since $f(x) \in B$, by (3) there exists a $\#rg$-open set A in X containing x such that $f(A) \subset B$. It follows that $x \in A \subset f^{-1}(B)$. Hence $f^{-1}(B)$ is $\#rg$-open.
(2) $\Rightarrow$ (1) Follows from the previous theorem.
(2) $\Rightarrow$ (4) Let A be any subset of X. Let $y \notin kerf(A)$. Then there exists a closed set F containing y such that $f(A) \cap F = \phi$. Hence we have $A \cap f^{-1}(F) = \phi$. Hence, we obtain $f(\#rg\text{-}cl(A)) \cap F = \phi$ and $\#rg\text{-}clA \cap f^{-1}(F) = \phi$ and $y \notin f(\#rg\text{-}cl(A))$. Thus $f(\#rg\text{-}cl(A)) \subset ker(f(A))$.
(4) $\Rightarrow$ (5) Let B be any subset of Y. By (4), $f(\#rg\text{-}cl(f^{-1}(B))) \subset ker(B)$ and $\#rg\text{-}cl(f^{-1}(B)) \subset f^{-1}(ker(B))$
(5) $\Rightarrow$ (1) Let B be any open set of Y. By (5), $\#rg\text{-}cl(f^{-1}(B)) \subset f^{-1}(ker(B)) = f^{-1}(B)$ and $\#rg\text{-}cl(f^{-1}(B)) = f^{-1}(B)$. We obtain that, $f^{-1}(B)$ is $\#rg$-closed in X. ■

Theorem 2.4. Suppose that X and Y are spaces and $\#RGO(X)$ is closed under arbitrary unions. If a function $f : X \to Y$ is contra $\#rg$-continuous and Y is regular, then f is $\#rg$-continuous.

Proof. Let x be an arbitrary point of X and V be an open set of Y containing $f(x)$. Since Y is regular, there exist an open set G in Y containing $f(x)$ such that $cl(G) \subset V$. Since f is contra $\#rg$-continuous, there exist $U \in \#RGO(X)$ containing x such that $f(U) \subset cl(G)$. Then $f(U) \subset cl(G) \subset V$. Hence, f is $\#rg$-continuous. ■

Theorem 2.5. Let $f : (X, T) \to (Y, S)$ be a function. Suppose that (X, T) is a $\#rg$-$T_{\frac{1}{2}}$ space. Then the following are equivalent:

(1) f is contra $\#rg$-continuous,

(2) f is contra continuous.

Proof. The proof is obvious. ■

Theorem 2.6. If a function $f : X \to \prod Y_i$ is contra $\#rg$-continuous, then $p_i \circ f : X \to Y_i$ is contra $\#rg$-continuous for each $i \in I$, where p_i is the projection of $\prod Y_i$ onto Y_i.

Proof. Let V_i be any open set of Y_i. Since p_i is continuous, $p_i^{-1}(V_i)$ is open in $\prod Y_i$. Since f is contra $\#rg$-continuous, $f^{-1}(p_i^{-1}(V_i)) = (p_i \circ f)^{-1}(V_i) \in \#RGC(X)$. This shows that $p_i \circ f$ is contra $\#rg$-continuous for each $i \in I$. ■

Definition 2.7. A topological space (X, T) is said to be **locally $\#rg$-indiscrete** if every $\#rg$-open set of X is closed in X.

Theorem 2.8. Let $f : (X,\ T) \to (Y,\ S)$ be a function. If f is contra #rg-continuous and $(X,\ T)$ is locally #rg-indiscrete, then f is continuous.

Definition 2.9. A function $f : X \to Y$ is said to be **#rg-irresolute** if $f^{-1}(V) \in \#RGO(X)$ for each $V \in \#RGO(Y)$.

Theorem 2.10. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then, the following properties hold:

(1) If f is #rg-irresolute and g is contra #rg-continuous, then $g \circ f : X \to Z$ is contra #rg-continuous.

(2) If f is contra #rg-continuous and g is continuous, then $g \circ f : X \to Z$ is contra #rg-continuous.

(3) If f is contra #rg-continuous and g is RC-continuous, then $g \circ f : X \to Z$ is #rg-continuous.

(4) If f is #rg-continuous and g is contra continuous, then $g \circ f : X \to Z$ is contra #rg-continuous.

Theorem 2.11. Suppose that $\#RGC(Y)$ is closed under arbitrary intersections. If $f : X \to Y$ is a surjective #rg-open function and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is contra #rg-continuous, then g is contra #rg-continuous.

Proof. Suppose that x and y are two points in X and Y respectively, such that $f(x) = y$. Let $B \in C(Z, (g \circ f)(x))$. Then there exits a #$rg$-open set A in X containing x such that $g(f(A)) \subset B$. Since f is #rg-open, $f(A)$ is a #rg-open in Y containing y such that $g(f(A)) \subset B$. This implies that g is contra #rg-continuous. ∎

Corollary 2.12. Let $f : X \to Y$ be a surjective #rg-irresolute and #rg-open function and let $g : Y \to Z$ be a function. Suppose that $\#RGC(Y)$ is closed under arbitrary intersections. Then $g \circ f : X \to Z$ is contra #rg-continuous if and only if g is contra #rg-continuous.

Proof. Follows from Theorem 2.10 and 2.11 ∎

3. Properties of contra #rg-continuous functions

Definition 3.1. The #rg-frontier of a subset A of a space X is given by $\#rg\text{-}fr(A) = \#rg\text{-}cl(A) \cap \#rg\text{-}cl(X \backslash A)$.

Theorem 3.2. Let the collection of all #rg-closed sets of a space $(X,\ \tau)$ be closed under arbitrary intersections. The set of all points $x \in X$ at which a function $f : (X, \tau) \to (Y, \sigma)$ is not contra #rg-continuous is identical with the union of #rg-frontier of the inverse images of closed sets containing $f(x)$.

Proof. ($\Rightarrow$) Suppose that f is not contra $\#rg$-continuous at $x \in X$. Then there exists a closed set A of Y containing $f(x)$ such that $f(U)$ is not contained in A for every $U \in \#RGO(X)$ containing x. Then $U \cap (X \backslash f^{-1}(A)) \neq \phi$ for every $U \in \#RGO(X)$ containing x and hence $x \in \#rg\text{-}cl(X \backslash f^{-1}(A))$. On the other hand, we have $x \in f^{-1}(A) \subset \#rg\text{-}cl(f^{-1}(A))$ and hence $x \in \#rg\text{-}fr(f^{-1}(A))$.

($\Leftarrow$) Suppose that f is contra $\#rg$-continuous at $x \in X$, and let A be a closed set of Y containing $f(x)$. Then there exists $U \in \#RGO(X)$ containing x such that $U \subset f^{-1}(A)$; hence $x \in \#rg\text{-}int(f^{-1}(A))$. Therefore, $x \notin \#rg\text{-}fr(f^{-1}(A))$ for each closed set A of Y containing $f(x)$. This completes the proof. ■

Corollary 3.3. Let $\#RGC(X)$ be closed under arbitrary intersections. A function $f : X \to Y$ is not contra $\#rg$-continuous at x if and only if $x \in \#rg\text{-}fr(f^{-1}(F))$ for some $F \in C(Y, f(x))$.

Definition 3.4. A space X is said to be $\#rg\text{-}T_1$ if for each pair of distinct points x and y in X, there exists $\#rg$-open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

Definition 3.5. A space X is said to be $\#rg\text{-}T_2$ if for each pair of distinct points x and y in X, there exists $U \in \#RGO(X, x)$ and $V \in \#RGO(X, y)$ such that $U \cap V = \phi$.

Theorem 3.6. Let X and Y be topological spaces. If

(1) for each pair of distinct points x and y in X, there exists a function f of X into Y such that $f(x) \neq f(y)$.

(2) Y is an Urysohn space and f is contra $\#rg$-continuous at x and y, then X is $\#rg\text{-}T_2$.

Proof. Let x and y be distinct points in X. Then, there exists a Urysohn space Y and a function $f : X \to Y$ such that $f(x) \neq f(y)$ and f is contra $\#rg$-continuous at x and y.

Let $z = f(x)$ and $v = f(y)$. Then $z \neq v$. We have to prove X is $\#rg\text{-}T_2$ space. Since Y is Urysohn, there exist open sets V and W containing z and v, respectively such that $cl(V) \cap cl(W) = \phi$. Since f is contra $\#rg$-continuous at x and y, then there exists $\#rg$-open sets A and B containing x and y, respectively such that, $f(A) \subset cl(V)$ and $f(B) \subset cl(W)$. We have $A \cap B = \phi$. Since $cl(V) \cap cl(W) = \phi$, Hence X is $\#rg\text{-}T_2$. ■

Corollary 3.7. Let $f : X \to Y$ be a contra $\#rg$-continuous injection. If Y is an Urysohn space, then it is $\#rg\text{-}T_2$.

Definition 3.8. A space X is said to be $\#rg$-connected if X is not the union of two disjoint nonempty $\#rg$-open sets.

Theorem 3.9. For a topological space X, the following properties are equivalent:

(1) X is $\#rg$-connected,

(2) The only subsets of X which are both $\#rg$-open and $\#rg$-closed are the empty set ϕ and X,

(3) Each contra $\#rg$-continuous function of X into a discrete space Y with at least two points is a constant function.

Proof. (1) $\Rightarrow$ (2) Suppose $A \subset X$ is a proper subset which is both $\#rg$-open and $\#rg$-closed. Then its complement $X \backslash A$ is also $\#rg$-open and $\#rg$-closed. Then $X = A \cup (X \backslash A)$ is a disjoint union of two nonempty $\#rg$-open sets which contradicts the fact that X is $\#rg$-connected. Hence, $A = \phi$ or X.

(2) $\Rightarrow$ (1) Suppose $X = A \cup B$ where $A \cap B = \phi$, $A \neq \phi$, $B \neq \phi$ and A and B are $\#rg$-open. Since $A = X \backslash B$, A is $\#rg$-closed. But by hypothesis $A = \phi$, which is a contradiction. Hence (1) holds.

(2) $\Rightarrow$ (3) Let $f : X \to Y$ be a contra $\#rg$-continuous function where Y is a discrete space with atleast two points. Then $f^{-1}(\{y\})$ is $\#rg$-closed and $\#rg$-open for each $y \in Y$ and $X = \cup\{f^{-1}(\{y\}) : y \in Y\}$. By hypothesis, $f^{-1}(\{y\}) = \phi$ or X. If $f^{-1}(\{y\}) = \phi$ for all $y \in Y$, f is not a function. Also there cannot exist more than one $y \in Y$ such that $f^{-1}(\{y\}) = X$. Hence, there exists only one $y \in Y$ such that $f^{-1}(\{y\}) = X$ and $f^{-1}(\{y_1\}) = \phi$ where $y \neq y_1 \in Y$. This shows that f is a constant function.

(3) $\Rightarrow$ (2) Let P be both $\#rg$-open and $\#rg$-closed in X. Suppose $P \neq \phi$. Let $f : X \to Y$ be a contra $\#rg$-continuous function defined by $f(P) = \{a\}$ and $f(X \backslash P) = \{b\}$ where $a \neq b$ and $a,\ b \in Y$. By hypothesis, f is costant. Therefore, $P = X$. ■

Theorem 3.10. If f is a contra $\#rg$-continuous function from a $\#rg$-connected space X onto any space Y, then Y is not a discrete space.

Proof. Suppose that Y is discrete. Let A be a proper nonempty clopen subset of Y. Then $f^{-1}(A)$ is a proper nonempty $\#rg$-clopen subset of X, which is a contradiction to the fact that X is $\#rg$-connected. ■

Theorem 3.11. A space X is $\#rg$-connected if every contra $\#rg$-continuous function from a space X into any T_0-space Y is constant.

Proof. Suppose that X is not $\#rg$-connected and that every contra $\#rg$-continuous function from X into Y is constant. Since X is not $\#rg$-connected, there exists a proper nonempty $\#rg$-clopen subset A of X. Let $Y = \{a,\ b\}$ and $\tau = \{Y,\ \phi,\ \{a\},\ \{b\}\}$ be a topology for Y. Let $f : X \to Y$ be a function such that $f(A) = \{a\}$ and $f(X \backslash A) = \{b\}$. Then f is non-constant and contra $\#rg$-continuous such that Y is T_0, which is a contradiction. Hence, X must be $\#rg$-connected. ■

Theorem 3.12. If $f : X \to Y$ is a contra $\#rg$-continuous surjection and X is $\#rg$-connected, then Y is connected.

Proof. Suppose that Y is not a connected space. There exists nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y. Since f is contra $\#rg$-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are $\#rg$-open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not $\#rg$-connected. This contradicts that Y is not connected assumed. Hence, Y is connected. ■

Theorem 3.13. Let $p : X \times Y \to X$ be a projection. If A is $\#rg$-closed subset of X, then $p^{-1}(A) = A \times Y$ is $\#rg$-closed subset of $X \times Y$.

Proof. Let $A \times Y \subset U$ and U be rw-open subset of $X \times Y$. Then $U = V \times Y$ for some rw-open set of X. Since A is $\#rg$-closed in X, $cl(A) \subset V$ and so $cl(A) \times Y \subset V \times Y = U$, ie., $cl(A \times Y) \subset U$. Hence, $A \times Y$ is $\#rg$-closed subset of $X \times Y$. ■

Proposition 3.14. If $f : X \to Y$ is a $\#rg$-irresolute surjection and X is $\#rg$-connected, then Y is $\#rg$-connected.

Proposition 3.15. If the product space of two nonempty topological spaces is $\#rg$-connected, then each factor space is $\#rg$-connected.

Proof. Let $X \times Y$ be the product space of the nonempty spaces X and Y and $X \times Y$ be $\#rg$-connected. The projection $p : X \times Y \to X$ is $\#rg$-irresolute and then $p(X \times Y) = X$ is $\#rg$-connected. The proof for the space Y is similar to the case of X. ■

Definition 3.16. A space X is said to be

(i) $\#rg$-compact if every $\#rg$-open cover of X has a finite subcover,

(ii) countably $\#rg$-compact if every countable cover of X by $\#rg$-open sets has a finite subcover,

(iii) $\#rg$-Lindelof if every $\#rg$-open cover of X has a countable subcover.

Theorem 3.17. If $f : X \to Y$ is contra $\#rg$-continuous and A is $\#rg$-compact relative to X, then $f(A)$ is strongly S-closed in Y.

Proof. Let $\{V_i : i \in I\}$ be any cover of $f(A)$ by closed sets of the subspace $f(A)$. For each $i \in I$, there exists a closed set A_i of Y such that $V_i = A_i \cap f(A)$. For each $x \in A$, there exists $i(x) \in I$ such that $f(x) \in A_{i(x)}$ and there exists $U_x \in \#RGO(X, x)$ such that $f(U_x) \subseteq A_{i(x)}$. Since the family $\{U_x : x \in A\}$ is a cover of A by $\#rg$-open sets of X, there exists a finite subset A_0 of A such that $A \subseteq \cup\{U_x : x \in A_0\}$. Hence, we obtain $f(A) \subseteq \cup\{f(U_x) : x \in A_0\}$ which is a subset of $\cup\{A_{i(x)} : x \in A_0\}$. Thus, $f(A) = \cup\{V_{i(x)} : x \in A_0\}$ and hence $f(A)$ is strongly S-closed. ■

Corollary 3.18. If $f : X \to Y$ is contra $\#rg$-continuous surjection and X is $\#rg$-compact, then Y is strongly S-closed.

Theorem 3.19. If the product space of two nonempty topological spaces is $\#rg$-compact, then the factor space is $\#rg$-compact.

Proof. Let $X \times Y$ be the product space of the nonempty spaces X and Y and $X \times Y$ be $\#rg$-compact. The projection $p : X \times Y \to X$ is $\#rg$-irresolute and then $p(X \times Y) = X$ is $\#rg$-compact. The proof for the space Y is similar to the case of X. ■

Theorem 3.20. The contra $\#rg$-continuous images of $\#rg$-lindelof (resp. countably $\#rg$-compact) spaces are strongly S-lindelof (respectively strongly countably S-closed).

Proof. Let $f : X \to Y$ be a contra $\#rg$-continuous surjection. Let $\{V_i : i \in I\}$ be any closed cover of Y. Since f is contra $\#rg$-continuous, then $f^{-1}(V_i) : i \in I\}$ is a $\#rg$-open cover of X and hence there exists a countable subset I_0 of I such that $X = \cup\{f^{-1}(V_i) : i \in I_0\}$. Therefore, we have $Y = \cup\{V_i : i \in I_0\}$ and Y is strongly S-Lindelof. ■

Definition 3.21. The graph $G(f)$ of a function $f : X \to Y$ is said to be contra $\#rg$-graph if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist a $\#rg$-open set A in X containing x and a closed set B in Y containing y such that $(A \times B) \cap G(f) = \phi$.

Proposition 3.22. The following properties are equivalent for the graph $G(f)$ of a function f:

(i) $G(f)$ is contra $\#rg$-graph;

(ii) for each $(x, y) \in (X \times Y)\backslash G(f)$, there exists a $\#rg$-open set A in X and a closed set B in Y containing y such that $f(A) \cap B = \phi$.

Theorem 3.23. If $f : X \to Y$ is contra $\#rg$-continuous and Y is Urysohn, $G(f)$ is contra-$\#rg$-graph in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. It follows that $f(x) \neq y$. Since Y is Urysohn, there exists open sets B and C such that $f(x) \in B$, $y \in C$ and $cl(B) \cap cl(C) = \phi$. Since f is contra $\#rg$-continuous, there exists a $\#rg$-open set A in X containing x such that $f(A) \subseteq cl(B)$. Therefore, $f(A) \cap cl(C) = \phi$ and $G(f)$ is contra-$\#rg$-graph in $X \times Y$. ■

Theorem 3.24. Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra $\#rg$-continuous, then f is contra-$\#rg$-continuous.

Proof. Let U be an open set in Y, then $X \times U$ is an open set in $X \times Y$. It follows that $f^{-1}(U) = g^{-1}(X \times U) \in |\#RGC(X)$. Thus f is contra-$\#rg$-continuous. ■

Theorem 3.25. If $f : X \to Y$ and $g : X \to Y$ are contra $\#rg$-continuous and Y is Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is $\#rg$-closed in X.

Proof. Let $x \in X\backslash E$. Then $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets V and C such that $f(x) \in V$, $g(x) \in C$ and $cl(V) \cap cl(C) = \phi$. Since f and g are contra $\#rg$-continuous, $f^{-1}(cl(V)) \in \#RGO(X)$ and $g^{-1}(cl(C)) \in \#RGO(X)$. Let

$U = f^{-1}(cl(V))$ and $G = g^{-1}(cl(C))$. Then U and V contain x. Set $A = U \cap G$. A is $\#rg$-open in X. Hence $f(A) \cap g(A) = \phi$ and $x \notin \#rg\text{-}cl(E)$. Thus, E is $\#rg$-closed in X. ∎

Definition 3.26. A subset A of a topological space X is said to be $\#rg$-dense in X if $\#rg\text{-}cl(A) = X$.

Theorem 3.27. Let $f : X \to Y$ and $g : X \to Y$ be functions. If

(1) Y is Urysohn,

(2) f and g are contra $\#rg$-continuous,

(3) $f = g$ on $\#rg$-dense set $A \subset X$, then $f = g$ on X.

Proof. Since f and g are contra $\#rg$-continuous and Y is Urysohn, by the previous theorem $E = \{x \in X : f(x) = g(x)\}$ is $\#rg$-closed in X. We have $f = g$ on $\#rg$-dense set $A \subset X$. Since $A \subset E$ and A is $\#rg$-dense set in X, then $X = \#rg\text{-}cl(A) \subset \#rg\text{-}cl(E) = E$. Hence, $f = g$ on X. ∎

Definition 3.28. A space X is said to be weakly Hausdorff [9] if each element of X is an intersection of regular closed sets.

Theorem 3.29. If $f : X \to Y$ is a contra $\#rg$-continuous injection and Y is weakly Hausdorff, then X is $\#rg$-T_1.

Proof. Suppose that Y is weakly Hausdorff. For any distinct points x and y in X, there exists regular closed sets A, B in Y such that $f(x) \in A$, $f(y) \notin A$, $f(x) \notin B$ and $f(y) \in B$. Since f is contra $\#rg$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are $\#rg$-open subsets of X such that $x \in f^{-1}(A)$, $y \notin f^{-1}(A)$, $x \notin f^{-1}(B)$ and $y \in f^{-1}(B)$. This shows that X is $\#rg$-T_1. ∎

Theorem 3.30. Let $f : X \to Y$ have a contra-$\#rg$-graph. If f is injective, then X is $\#rg$-T_1.

Proof. Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in (X \times Y)\backslash G(f)$. Then, there exist a $\#rg$-open set U in X containing x and $F \in C(Y, f(y))$ such that $f(U) \cap F = \phi$; hence $U \cap f^{-1}(F) = \phi$. Therefore, we have $y \notin U$. This implies that X is $\#rg$-T_1. ∎

Theorem 3.31. Let $f : X \to Y$ be a contra-$\#rg$-continuous injection. If Y is an ultra Hausdorff space, then X is $\#rg$-T_2.

Proof. Let x and y be any two distinct points in X. Then, $f(x) \neq f(y)$ and there exist clopen sets A and B containing $f(x)$ and $f(y)$, respectively such that $A \cap B = \phi$. Since f is contra $\#rg$-continuous, then $f^{-1}(A) \in \#RGO(X)$ and $f^{-1}(B) \in \#RGO(X)$ such that $f^{-1}(A) \cap f^{-1}(B) = \phi$. Hence, X is $\#rg$-T_2. ∎

Definition 3.32. A topological space X is said to be #rg-normal if each pair of nonempty disjoint closed sets can be separated by disjoint #rg-open sets.

Theorem 3.33. If $f : X \to Y$ is a contra-#rg-continuous closed injection and Y is ultra normal, then X is #rg-normal.

Proof. Let A and B be disjoint closed subsets of X. Since f is a closed injection, $f(A)$ and $f(B)$ are disjoint and closed in Y. Since Y is ultra normal, $f(A)$ and $f(B)$ are separated by disjoint clopen sets C and D, respectively. Thus, $A \subseteq f^{-1}(C)$, $B \subseteq f^{-1}(D) \in \#RGO(X)$ and $f^{-1}(C) \cap f^{-1}(D) = \phi$. Hence, X is #rg-normal. ■

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