Global Journal of Mathematical Sciences: Theory and Practical
ISSN 0974-3200 Volume 6, Number 2 (2014), pp. 101-109

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Algebraic Rational Cubic Spline with Constrained Control

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ABSTRACT

In this paper a rational cubic algebraic spline with two shape parameters is developed to create a high-order smoothness interpolation using function values and derivative values which are being interpolated. This is a kind of rational cubic interpolation with quadratic denominator. This rational spline interpolant is monotonic interpolant to given monotonic data. The more important achievement is that it is also used to constrain the shape of the interpolant curve such as to force it to be in the given region, all because of the selectable parameters in the rational spline itself.

Keywords: Interpolation, Rational spline, Shape parameters, Monotonicity, Error estimation, Constrained interpolation.

1. Introduction

In recent years spline methods have become the main tools for solving a large number of problems involving interpolation and approximation of functions. In many industrial problems of design and manufacturing, it is usually needed to generate a smooth function which passes through the given data set and preserves shape properties like positivity, monotonicity and convexity. Stability of radioactive substance and chemical reactions, solvability of solute in solvent, population statistics [2], resistance offered by an electric circuit, probability distribution are a few examples of entities which need to be always positive. Monotonicity is applied in the specification of Digital to Analog Converters (DACs), Analog to Digital Converters (ADCs) and sensors. These devices are used in control system applications where as non-monotonicity is unacceptable [6]. Erythrocyte sedimentation rate (E.S.R.) in cancer patients, uric acid level in patients suffering from gout, data generated from stress of a material [6], approximation of couples and quasi couples in statistics, rate of dissemination of drug in blood [1] are a few examples of entities which are always required to be monotone. Many authors have contributed to the problems of shape preserving interpolation [4, 5, 7, 9, 12].

Rational functions have become more popular in the construction of curves and surfaces in CAD, CAM because of their more flexibility and shape preserving ability [3, 5, 6, 10]. It has been observed that many simple shapes including conic section and quadric surfaces can not be represented exactly by piecewise polynomials, whereas rational polynomials can exactly represent all conic sections and quadric surfaces in an easy manner [8].

In section 2 we have constructed C^1 rational cubic interpolant with two shape parameters. In Section 3 we have derived the sufficient conditions for monotonicity of the interpolant. Section 4 is about error estimation of the interpolant. This construction confirms that the expected approximation order is $O(h^2)$. The visualization problems of constrained data interpolation is discussed in section 5. In section 6 we have discussed a numerical example with graphical representation for constrained data interpolation.

2. C^1 Rational Cubic Spline Interpolant

Let $\{(t_i, f_i), i = 1, 2, \dots, n\}$ be a given set of data point defined over the interval $[a, b]$ where a = $t_1 < t_2 < \dots < t_n$ = b and f_i are the function values at the knots.

Let $h_i = t_{i+1} - t_i$, $\Delta_i = \frac{f_{i+1} - f_i}{h_i}$, $i = 1, 2, \dots, n-1$.

For $t \in [t_i, t_{i+1}]$, let $\theta = \frac{t - t_i}{h_i}$, $0 \le \theta \le 1$.

A piecewise rational cubic function $S(t) \in c^1[t_1, t_n]$ and is defined for $t \in [t_i, t_{i+1}]$ as:

$$S(t) = S_i(t) = \frac{P_i(t)}{Q_i(t)} = \frac{(1-\theta)^2\alpha_i A_i + \theta^2(1-\theta)B_i + \theta(1-\theta)^2 C_i + \theta^2\beta_i D_i}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i} \tag{1}$$

The rational cubic interpolant has the following interpolatory properties:

$$S(t_i) = f_i, S(t_{i+1}) = f_{i+1}, S(t_i) = d_i, S(t_{i+1}) = d_{i+1} \tag{2}$$

which provides the following manipulations:

$$A_i = f_i, B_i = f_{i+1} - \beta_i h_i d_{i+1}, C_i = f_i + \alpha_i h_i d_i, D_i = f_{i+1},$$

where d_i are the derivative values of the interpolant at the knots and α_i, β_i are positive shape parameters. In most applications, derivative parameters d_i are not given and hence must be determined from data (t_i, f_i). An appropriate choice is mentioned here (see [13]):

$$d_i = \frac{h_i\Delta_{i-1} + h_{i-1}\Delta_i}{h_i + h_{i-1}}, i = 2, 3, \dots, n-1,$$

$$d_1 = \Delta_1 + \frac{(\Delta_1 - \Delta_2)h_1}{h_1 + h_2}$$

$$d_n = \Delta_{n-1} + \frac{(\Delta_{n-1} - \Delta_{n-2})h_{n-1}}{h_{n-1} + h_{n-2}} \tag{3}$$

This method is based on three point difference approximation for the d_i.

3. Monotonicity-Preserving Rational Spline Interpolation

We assume a monotonic increasing data, so that

$$f_1 < f_2 < \cdots < f_n \text{ or equivalently } \Delta_i > 0 \ (i = 1, 2, \ldots, n-1) \tag{4}$$

To have a monotonic interpolant $S(t)$, it is necessary that the derivative parameters d_i should satisfy:

$$d_i > 0 \ (d_i < 0 \text{ for monotonic decreasing data}) \ i = 1, 2, \ldots, n \tag{5}$$

Now $S(t)$ is monotonically increasing if and only if $S(t) > 0, t \in [t_i, t_{i+1}]$. After some simplification it can be written as

$$S(t) = \frac{X + [\sum_{i=1}^{5} R_i \, \theta^{i-1}(1-\theta)^{5-i}]}{[Q_i(\theta)]^2} \tag{6}$$

where

$$X = \theta^3(1-\theta)^2 K_1 + \theta^2(1-\theta)^3 K_2 + \theta^2(1-\theta)K_3 + \theta(1-\theta)^2 K_4$$
$$K_1 = K_2 = (B_i - C_i),$$
$$K_3 = K_4 = 2\alpha_i\beta_i(D_i - A_i),$$
$$R_1 = \alpha_i(C_i - A_i),$$
$$R_2 = 2\alpha_i(B_i - A_i),$$
$$R_3 = [\alpha_i(B_i - A_i) + \beta_i(D_i - C_i)],$$
$$R_4 = 2\beta_i(D_i - C_i),$$
$$R_5 = \beta_i(D_i - B_i), \tag{7}$$

After simple manipulation above equations can be written as

$$K_1 = K_2 = \Delta_i - \beta_i d_{i+1} - \alpha_i d_i,$$
$$K_3 = K_4 = 2\alpha_i\beta_i(D_i - A_i),$$
$$R_1 = \alpha_i{}^2 d_i,$$
$$R_2 = 2\alpha_i(\Delta_i - \beta_i d_{i+1}),$$
$$R_3 = [\alpha_i(\Delta_i - \beta_i d_{i+1}) + \beta_i(\Delta_i - \alpha_i d_i)],$$
$$R_4 = 2\beta_i(\Delta_i - \alpha_i d_i),$$
$$R_5 = \beta_i{}^2 d_{i+1},$$

We observe that for $\alpha_i, \beta_i > 0$ the denominator of rational function $S(t)$ given in (6) is positive.

Therefore considering the numerator in (6), we find that $S(t)$ is positive if $R_2 > 0$ and $R_4 > 0$ for monotonic increasing data. Thus the sufficient condition that $R_2 > 0$ and $R_4 > 0$ is

$$\alpha_i < \frac{\Delta_i}{d_i} \text{ and } \beta_i < \frac{\Delta_i}{d_{i+1}} \tag{9}$$

Therefore $S(t)$ is positive if (9) holds. We have thus proved the following :

Theorem 3.1 Given a monotonic increasing set of data satisfying (4) and the derivative values satisfying (5), there exists a monotone rational spline interpolant $S(t) \in C^1[t_{1,} t_n]$ involving the shape parameters α_i, and β_i which satisfies the interpolatory conditions (2) provided (9) holds.

4. Error Estimation

In this Section, the error of interpolation is estimated when the function $f(t)$ being interpolated is $C^2[t_{1,} t_n]$. The interpolation scheme developed in this section is local, which allows investigating the error in an arbitrary subinterval $I_i = [t_i, t_{i+1}]$ without loss of generality. Using Peano Kernel Theorem [11] the error of interpolation in each subinterval $I_i = [t_i, t_{i+1}]$ is:

$$R[f] = f(t) - S(t) = \frac{1}{2} \int_{t_i}^{t_{i+1}} f^2(\tau) R_t[(t-\tau)_+] d\tau, \tag{10}$$

It is assumed that the function being interpolated is $f(t) \in C^2[t_{1,} t_n]$. The absolute error in $I_i = [t_i, t_{i+1}]$ is:

$$f(t) - S(t) \le \frac{1}{2} \|f^2(\tau)\| \int_{t_i}^{t_{i+1}} R_t[(t-\tau)_+] \, d(\tau), \tag{11}$$

with

$$lR_t [(t-\tau)_+] = \begin{cases} p(\tau) \; t_i < \tau < t \\ q(\tau) \; t < \tau < t_{i+1} \end{cases} \tag{12}$$

where $p(\tau)$, $q(\tau)$ are rational polynomials given by

$$p(\tau) = (t-\tau) - \left[\frac{[(\theta^2(1-\theta)+\theta^2\beta_i)(t_{i+1}-\tau)] - \beta_i h_i \theta^2(1-\theta)}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i}\right],$$

$$q(\tau) = -\left[\frac{[(\theta^2(1-\theta)+\theta^2\beta_i)(t_{i+1}-\tau)] - \beta_i h_i \theta^2(1-\theta)}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i}\right].$$

Thus,

$$\int_t^{t_{i+1}} R_t[(t-\tau)_+] \, d\tau = \int_{t_i}^{t} p(\tau) \, d\tau + \int_t^{t_{i+1}} q(\tau) \, d\tau \tag{13}$$

For $p(\tau)$, since

$$p(t_i) = h_i \left[\frac{\theta(1-\theta)^2\alpha_i}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i}\right] \ge 0,$$

$$p(t) = -h_i \left[\frac{\theta^2(1-\theta)^2}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i}\right] \le 0.$$

Thus, it may be observed that, there is a zero point $p(\tau_1)$ of $p(\tau)$ in $[t_i, t_{i+1}]$ given by

$$\tau_1 = t_{i+1} - \left[\frac{h_i(1-\theta)\{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2\}}{(1-\theta)^2\alpha_i + \theta(1-\theta)^2}\right].$$

Thus,

$$\int_{t_i}^{t} p(\tau)\, d\tau \;=\; \int_{t_i}^{\tau_1} p(\tau)\, d\tau + \int_{\tau_1}^{t} -p(\tau)\, d\tau = h_i{}^2 M \tag{14}$$

where

$$M = M_1 + \frac{\theta^2(1-\theta) + \theta^2\beta_i}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i} M_2 + \frac{\beta_i\theta^2(1-\theta)}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i} M_3$$

with

$$M_1 = \frac{\theta^2}{2} - \left[\frac{\theta^2(1-\theta)^2}{(1-\theta)^2\alpha_i + \theta(1-\theta)^2}\right]^2,$$

$$M_2 = \left(\frac{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2}{(1-\theta)^2\alpha_i + \theta(1-\theta)^2}\right)^2 - \frac{1}{2} - \frac{(1-\theta)^2}{2},$$

$$M_3 = \frac{\theta(1-\theta)^2\alpha_i - \theta^2(1-\theta)^2}{(1-\theta)^2\alpha_i + \theta(1-\theta)^2}.$$

For $q(\tau)$, since

$$q(t) = -\left[\frac{h_i(1-\theta)\left(\theta^2(1-\theta) + \theta^2\beta_i\right) - \beta_i h\theta^2(1-\theta)}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i}\right] \le 0,$$

$$q(t_{i+1}) = \left[\frac{\beta_i h_i\theta^2(1-\theta)}{(1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i}\right] \le 0.$$

Let τ_2 is the zero point(root) of $q(\tau)$ then, it may easily be seen that

$$\tau_2 = t_{i+1} - \frac{\beta_i h_i \theta^2(1-\theta)}{\theta^2(1-\theta) + \theta^2\beta_i}$$

and

$$\int_{t}^{t_{i+1}} q(\tau)\, d\tau = \int_{t}^{\tau_2} -q(\tau) d\tau + \int_{\tau_2}^{t_{i+1}} q(\tau) d\tau = h_i{}^2\, N \tag{15}$$

where

$$N = \frac{(1-\theta)^2\theta^4\left((1-\theta)^2 + \beta_i(1-\theta) + \beta_i{}^2\right)}{\theta^2(1-\theta) + \theta^2\beta_i}$$

The above discussion leads to the following manipulation:

$$f(t) - S(t) \le \frac{1}{2}\| f^2(\tau)\|\; h_i{}^2\, \omega(\alpha_i, \beta_i, \theta)$$

where

$$\omega(\alpha_i, \beta_i, \theta) = M + N.$$

The above discussion is summarized in the following theorem:

Theorem 4.1 The error of C^1 rational cubic function (1), for $f(t) \in C^2\,[t_1, t_n]$ in each subinterval is given by

$$f(t) - S(t) \le \frac{1}{2} \| f^2(\tau) \| \; {h_i}^2 \, c_i$$

where c_i is the maximum value of $\omega(\alpha_i, \beta_i, \theta)$ for $0 \le \theta \le 1$ and $\alpha_i, \beta_i > 0$

5. Constrained Data Interpolation

Let $\{(t_i, f_i), i = 1, 2, \dots, n\}$ be the given set of data points lying above the straight line $y = mt + c$
i.e.

$$f_i > mt_i + c \quad i = 1, 2, \dots, n. \tag{16}$$

The curve will lie above the straight line if the C^1 rational cubic function (1) satisfies the following condition:

$S(t) > mt + c$ (or $S(t) > u(t)$), $\forall\, t \in [t_0, t_n]$. For each subinterval $I_i = [t_i, t_{i+1}]$, the above relation in parametric form is expressed as

$$S(t) > u_i(1-\theta) + u_{i+1}\theta. \tag{17}$$

where $\theta = \frac{t - t_i}{h_i}$ and $u_i(1-\theta) + u_{i+1}\theta$ is the parametric equation of straight line with $u_i = mt_i + c$ and $u_{i+1} = mt_{i+1} + c$.

Since $Q_i(t) > 0$ for $t \in [t_i, t_{i+1}]$, then

$$S(t) = \frac{P_i(t)}{Q_i(t)} > u_i(1-\theta) + u_{i+1}\theta$$

Is equivalent to

$$P_i(t) - Q_i(t)[u_i(1-\theta) + u_{i+1}\theta] > 0$$

Let,

$$U_i(t) = P_i(t) - Q_i(t)[u_i(1-\theta) + u_{i+1}\theta], \tag{18}$$

it follows

$$U_i(t) = (1-\theta)^2 \alpha_i A_i + \theta^2(1-\theta)B_i + \theta(1-\theta)^2 C_i + \theta^2 \beta_i D_i$$

$$- ((1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i)(u_i(1-\theta) + u_{i+1}\theta) > 0 \tag{19}$$

Since,

$$((1-\theta)^2\alpha_i + \theta^2(1-\theta) + \theta(1-\theta)^2 + \theta^2\beta_i)(u_i(1-\theta) + u_{i+1}\theta)$$

$$= \alpha_i u_i(1-\theta)^2 + (u_i\beta_i + u_{i+1} - u_{i+1}\beta_i)\theta^2(1-\theta)$$

$$+ (u_i + u_i\beta_i + \alpha_i u_{i+1} - \alpha_i u_i)\theta(1-\theta)^2 + \beta_i u_{i+1}\theta^2,$$

equation (19) becomes

$$U_i(t) = (1-\theta)^2 \alpha_i Y_1 + \theta^2(1-\theta)Y_2 + \theta(1-\theta)^2 Y_3 + \theta^2 \beta_i Y_4 \tag{20}$$

where

$Y_1 = f_i - u_i,$

$Y_2 = (f_{i+1} - u_{i+1}) + \beta_i (u_{i+1} - u_i - h_i d_{i+1}),$

$Y_3 = (f_i - u_i) + \alpha_i (u_i - u_{i+1} + h_i d_i),$

$Y_4 = f_{i+1} - u_{i+1}.$

Now $U_i(t) > 0$ if $Y_i > 0$ for $i = 1, 2, 3, 4$. It is straightforward to know that $Y_1 > 0$ and $Y_4 > 0$ are true from the necessary condition defined in (16). If $Y_2 > 0$, $Y_3 > 0$ then $U_i(t) > 0$ for all $t \in [t_i, t_{i+1}]$.

Hence, the following theorem holds:

Theorem 5.1 The piecewise C^1 rational cubic interpolant $S(t)$, defined over the interval $[a, b]$, in (1), preserves the shape of data that lies above the straight line if in each subinterval $I_i = [t_i, t_{i+1}]$ the following sufficient conditions are satisfies

$$(f_i - u_i) + \alpha_i (u_i - u_{i+1} + h_i d_i) > 0, \tag{21}$$

$$(f_{i+1} - u_{i+1}) + \beta_i (u_{i+1} - u_i - h_i d_{i+1}) > 0.$$

For the case when

$$f_i < mt_i + c, i = 1, 2, \dots, n. \tag{22}$$

The curve will lie below the straight line if the C^1 rational cubic function (1) satisfies the following condition:

$S(t) < mt + c$ (or $S(t) < v(t)$), $\forall\, t \in [t_0, t_n]$. For each subinterval $I_i = [t_i, t_{i+1}]$, the above relation in parametric form is expressed as

$$S(t) < v_i(1 - \theta) + v_{i+1}\theta. \tag{23}$$

with $v_i = mt_i + c$ and $v_{i+1} = mt_{i+1} + c$.

Then the following theorem holds:

Theorem 5.2 The piecewise C^1 rational cubic interpolant $S(t)$, defined over the interval $[a, b]$, in (1), preserves the shape of data that lies below the straight line if in each subinterval *Ii* = [*ti; ti*+1] the following sufficient conditions are satisfies

$$(f_i - v_i) + \alpha_i (v_i - v_{i+1} + h_i d_i) < 0, \tag{21}$$

$$(f_{i+1} - v_{i+1}) + \beta_i (v_{i+1} - v_i - h_i d_{i+1}) < 0.$$

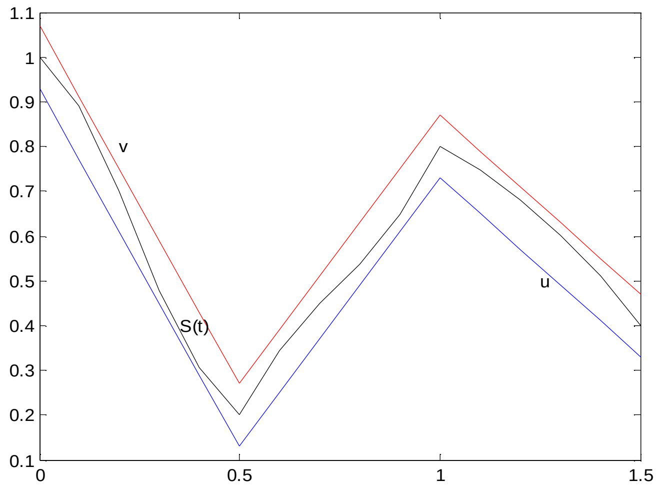

Figure 1: graph of v(t), S(t) and u(t) for data in Table 6.1

6. Numerical Example

The interpolating and constraining data and the parameters α_i and $\beta_i, i = 1, 2, 3$ are given in Table 6.1. It may be verified that both the given interpolating and constraining data and the parameters satisfy the relationship (16), (22) and the constraint inequalities (21), (24), so the interpolating curve defined by (1) must be bounded between u(t) and v(t) Fig. 1 shows that the interpolating curve S(t) is above u(t) and below v(t)

Table 6.1 The interpolating data, the constraining data and shape parameters

i	t_i	v_i	$f(t_i)$	u_i	α_i	β_i
1	0.0	1.07	1	0.93	0.001	1.994
2	0.5	0.27	0.2	0.13	1.151	1.19311
3	1.0	0.87	0.8	0.73	0.511	0.41905
4	1.5	0.47	0.4	0.33		

7. Conclusion

In this paper, the construction of a rational cubic spline with shape parameters α_i, β_i given, which is monotonic for monotonic data. The sufficient conditions for constraining the interpolating curves to be bounded between two given straight lines is derived and the approximation properties of the interpolant has been discussed, which is useful in the field of engineering.

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