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# Kähler manifold with a special type of semi-symmetric non-metric connection

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**Abstract**

In the present paper, we have obtained certain results for a Kähler manifold equipped with semi-symmetric non-metric connection and a special type of semi-symmetric non-metric connection. We have obtained the expressions for the curvature tensor, Ricci tensor and proved certain results related to them. We have also discussed the properties of torsion tensor as a second order parallel tensor.

**AMS subject classification:** 53C15.
**Keywords:** Semi-symmetric non-metric connection, Kähler manifold, Second order parallel tensor.

## 1. Introduction

The Riemannian manifold equipped with a semi-symmetric metric connection has been studied by O. C. Andonie [2], M. C. Chaki and A. Konar [3], U. C.De [4] etc., while a special type of semi-symmetric metric connection on a weakly symmetric Riemannian manifold has been studied by U. C. De and Joydeep Sengupta [5]. P. N. Pandey and S. K. Dubey [7] discussed an almost Grayan manifold admitting a semi-symmetric metric connection while a Kähler manifold equipped with semi-symmetric metric connection, a semi-symmetric non-metric connection on a Kähler manifold and an almost Hermitian manifold with semi-symmetric recurrent connection have been studied by P. N. Pandey and B. B. Chaturvedi [6, 8, 9]. Nirmala S. Agashe and Mangala R. Chafle [1] have studied semi-symmetric non-metric connection on a Riemannian manifold in 1992.

Let $Mn$ be an even dimensional differentiable manifold of differentiability class $C^{r+1}$. If there exists a vector valued linear function F of differentiability class $C^r$ such that for any vector field $X$

$$\overline{\overline{X}} + X = 0\,, \tag{1.1}$$

$$g\,(\overline{X}, \overline{Y}) = g(X, Y), \tag{1.2}$$

and

$$(D_X F)Y = 0\,, \tag{1.3}$$

where $\overline{X} = FX$, $g$ is non-singular metric tensor and $D$ is Riemannian connection, then $M_n$ is called a Kähler manifold.

We define another linear connection $\nabla$ for two arbitrary vector fields $X$ and $Y$ such that

$$\nabla_X Y = D_X Y + a\omega(X)Y + b\omega(Y)X\,, \tag{1.4}$$

where $\omega$ is a 1-form associated with an unit vector field $\rho$ which is parallel with respect to Riemannian connection $D$ and defined by $\omega(X) = g(X, \rho)$. Here $a$ and $b$ are non-zero real or complex numbers such that $a \neq b$.

Putting $g(Y, Z)$ in place of Y in(1.4), we haveu

$$(\nabla_X g)(Y, Z) = -a\omega(X)g(Y, Z) - b\omega(Y)g(X, Z) - b\omega(Z)g(X, Y)\,, \tag{1.5}$$

which shows that the connection $\nabla$ is non-metric.

## 2. Curvature tensor

From (1.4), we have

$$\nabla_Y Z = D_Y Z + a\omega(Y)Z + b\omega(Z)Y\,. \tag{2.1}$$

Replacing $Y$ for $\nabla_Y Z$ in equation (1.4), we have

$$\nabla_X \nabla_Y Z = D_X \nabla_Y Z + a\omega(X)\nabla_Y Z + b\omega(\nabla_Y Z)X\,. \tag{2.2}$$

Using (1.4) in (2.2), we have

$$\begin{aligned} \nabla_X \nabla_Y Z =& D_X D_Y Z + a(D_X\omega)(Y)Z + a\omega(D_X Y)Z + a\omega(Y)D_X Z \\ &+ b(D_X\omega)(Z)Y + b\omega(D_X Z)Y + b\omega(Z)D_X Y \\ &+ a\omega(X)D_Y Z + a^2\omega(X)\omega(Y)Z + ab\omega(X)\omega(Z)Y \\ &+ b\omega(D_Y Z)X + ab\omega(Y)\omega(Z)X + b^2\omega(Y)\omega(Z)X\,. \end{aligned} \tag{2.3}$$

Interchanging X and Y in the above equation, we get

$$\begin{aligned}\nabla_Y\nabla_X Z =& D_Y D_X Z + a(D_Y\omega)(X)Z + a\omega(D_Y X)Z + a\omega(X)D_Y Z\\&+ b(D_Y\omega)(Z)X + b\omega(D_Y Z)X + b\omega(Z)D_Y X\\&+ a\omega(Y)D_X Z + a^2\omega(Y)\omega(X)Z + ab\omega(Y)\omega(Z)X\\&+ b\omega(D_X Z)Y + ab\omega(X)\omega(Z)Y + b^2\omega(X)\omega(Z)Y\,.\end{aligned} \tag{2.4}$$

From equation (1.4), we may write

$$\nabla_{[X,Y]}Z = D_{[X,Y]}Z + a\omega([X,Y])Z + b\omega(Z)[X,Y]\,. \tag{2.5}$$

Subtracting (2.4) and (2.5) from (2.3), we have

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + b^2[\omega(X)Y - \omega(Y)X]\omega(Z)\,. \tag{2.6}$$

On a Kähler manifold the Riemannian curvature tensor satisfies the following

$$(i)\quad R(X,Y)\overline{Z} = \overline{R(X,Y)Z}\,, \tag{2.7}$$

$$(ii)\quad {}'R(X,Y,\overline{Z},\overline{W}) = {}'R(\overline{X},\overline{Y},Z,W)\,, \tag{2.8}$$

$$(iii)\quad {}'R(\overline{X},\overline{Y},\overline{Z},\overline{W}) = {}'R(X,Y,Z,W)\,, \tag{2.9}$$

$$(iv)\quad {}'R(X,Y,Z,\overline{W}) + {}'R(X,Y,\overline{Z},W) = 0\,, \tag{2.10}$$

$$(v)\quad {}'R(X,\overline{Y},\overline{Z},W) = {}'R(\overline{X},Y,Z,\overline{W})\,, \tag{2.11}$$

where ${}'R(X,Y,Z,W) = g(R(X,Y)Z,W)$ and ${}'\widetilde{R}(X,Y,Z,W) = g({}'\widetilde{R}(X,Y)Z,W)$. Therefore from (2.6), we have

$${}'\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + b^2[\omega(X)g(Y,W) - \omega(Y)g(X,W)]\omega(Z)\,. \tag{2.12}$$

Now, we propose:

**Theorem 2.1.** In a Kähler manifold equipped with the semi-symmetric non-metric connection $\nabla$, the curvature tensor satisfies the following:

$$(i)\qquad \overline{\widetilde{R}(X,Y)Z} + \overline{\widetilde{R}(Y,X)Z} = \widetilde{R}(X,Y)\overline{Z} + \widetilde{R}(Y,X)\overline{Z}\,, \tag{2.13}$$

$$(ii)\qquad {}'\widetilde{R}(X,Y,\overline{Z},\overline{W}) = {}'\widetilde{R}(\overline{X},\overline{Y},Z,W)\,, \tag{2.14}$$

if and only if

$$\begin{aligned}&[\omega(X)g(Y,\overline{W}) - \omega(Y)g(X,\overline{W})]\omega(\overline{Z})\\&\quad = [\omega(\overline{X})g(\overline{Y},W) - \omega(\overline{Y})g(\overline{X},W)]\omega(Z)\end{aligned}$$

$$(iii)\qquad {}'\widetilde{R}(X,Y,Z,\rho) = {}'R(X,Y,Z,\rho)\,, \tag{2.15}$$

$$(iv) \qquad {}'\widetilde{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = {}'\widetilde{R}(X, Y, Z, W)\,, \tag{2.16}$$

and only if

$$[\omega(\overline{X})g(Y, W) - \omega(\overline{Y})g(X, W)]\omega(\overline{Z}) = [\omega(X)g(Y, W) - \omega(Y)g(X, W)]\omega(Z).$$

$$(v) \qquad {}'\widetilde{R}(X, Y, Z, \overline{W}) + {}'\widetilde{R}(X, Y, \overline{Z}, W) = 0\,, \tag{2.17}$$

if and only if

$$[\omega(X)g(Y, \overline{W}) - \omega(Y)g(X, \overline{W})]\omega(Z) + [\omega(X)g(Y, W) - \omega(Y)g(X, W)]\omega(\overline{Z}) = 0.$$

$$(vi) \qquad {}'\widetilde{R}(X, \overline{Y}, \overline{Z}, W) = {}'\widetilde{R}(\overline{X}, Y, Z, \overline{W})\,, \tag{2.18}$$

if and only if

$$[\omega(X)g(\overline{Y}, W) - \omega(\overline{Y})g(X, W)]\omega(\overline{Z}) = [\omega(\overline{X})g(Y, \overline{W}) - \omega(Y)g(X, W)]\omega(Z).$$

*Proof.* Interchanging X and Y in (2.6), we get

$$\widetilde{R}(Y, X)Z = R(Y, X)Z + b^2[\omega(Y)X - \omega(X)Y]\omega(Z)\,. \tag{2.19}$$

Adding (2.6) and (2.19), we have

$$\widetilde{R}(X, Y)Z + \widetilde{R}(Y, X)Z = R(X, Y)Z + R(Y, X)Z\,. \tag{2.20}$$

Operating $F$ on both side of (2.20)and using (2.7), we get (2.13).
Barring Z and W in (2.12), we have

$${}'\widetilde{R}(X, Y, \overline{Z}, \overline{W}) = R(X, Y, \overline{Z}, \overline{W}) + b^2[\omega(X)g(Y, \overline{W}) - \omega(Y)g(X, \overline{W})]\omega(\overline{Z})\,. \tag{2.21}$$

Again barring X and Y in (2.12)

$$\begin{aligned} {}'\widetilde{R}(\overline{X}, \overline{Y}, Z, W) =& R(\overline{X}, \overline{Y}, Z, W) + b^2[\omega(\overline{X})g(\overline{Y}, W) \\ & - \omega(\overline{Y})g(\overline{X}, W)]\omega(Z)\,. \end{aligned} \tag{2.22}$$

Subtracting (2.22) from (2.21) and using (2.8), we get

$$\begin{aligned} {}'\widetilde{R}(X, Y, \overline{Z}, \overline{W}) - {}'\widetilde{R}(\overline{X}, \overline{Y}, Z, W) =& b^2[\omega(X)g(Y, \overline{W}) \\ & - \omega(Y)g(X, \overline{W})]\omega(\overline{Z}) - b^2[\omega(\overline{X})g(\overline{Y}, W) \\ & - \omega(\overline{Y})g(\overline{X}, W)]\omega(Z)\,. \end{aligned} \tag{2.23}$$

From (2.23) it is obvious that (2.14) holds if and only if

$$\begin{aligned} &[\omega(X)g(Y, \overline{W}) - \omega(Y)g(X, \overline{W})]\omega(\overline{Z}) \\ &\quad = [\omega(\overline{X})g(\overline{Y}, W) - \omega(\overline{Y})g(\overline{X}, W)]\omega(Z) \end{aligned}$$

Now Putting $W = \rho$ in (2.12), we get (2.15).

Barring X, Y, Z, W in (2.12) and using (1.2), we get

$$'\widetilde{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = R(\overline{X},\overline{Y},\overline{Z},\overline{W}) + b^2[\omega(\overline{X})g(Y,W) - \omega(\overline{Y})g(X,W)]\omega(\overline{Z})\,. \tag{2.24}$$

Subtracting (2.12) from (2.24)and using(2.9), we have

$$'\widetilde{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) - {'\widetilde{R}}(X,Y,Z,W) = b^2[\omega(\overline{X})g(Y,W) - \omega(\overline{Y})g(X,W)]\omega(\overline{Z}) - b^2[\omega(X)g(Y,W) - \omega(Y)g(X,W)]\omega(Z)\,. \tag{2.25}$$

From (2.25) it is obvious that (2.16) holds if and only if

$$[\omega(\overline{X})g(Y,W) - \omega(\overline{Y})g(X,W)]\omega(\overline{Z}) = [\omega(X)g(Y,W) - \omega(Y)g(X,W)]\omega(Z).$$

Barring $W$ in (2.12), we get

$$'\widetilde{R}(X,Y,Z,\overline{W}) = R(X,Y,Z,\overline{W}) + b^2[\omega(X)g(Y,\overline{W}) - \omega(Y)g(X,\overline{W})]\omega(Z)\,. \tag{2.26}$$

Barring Z in (2.12), we have

$$'\widetilde{R}(X,Y,\overline{Z},W) = R(X,Y,\overline{Z},W) + b^2[\omega(X)g(Y,W) - \omega(Y)g(X,W)]\omega(\overline{Z})\,. \tag{2.27}$$

In view of (2.26), (2.27) and (2.10), we can write

$$'\widetilde{R}(X,Y,Z,\overline{W}) + {'\widetilde{R}}(X,Y,\overline{Z},W) = b^2[\omega(X)g(Y,\overline{W}) - \omega(Y)g(X,\overline{W})]\omega(Z) + b^2[\omega(X)g(Y,W) - \omega(Y)g(X,W)]\omega(\overline{Z})\,. \tag{2.28}$$

From (2.28), we have (2.18).

Barring $Y$ and $Z$ in (2.12), we get

$$'\widetilde{R}(X,\overline{Y},\overline{Z},W) = R(X,\overline{Y},\overline{Z},W) + b^2[\omega(X)g(\overline{Y},W) - \omega(\overline{Y})g(X,W)]\omega(\overline{Z})\,. \tag{2.29}$$

Again barring $X$ and $W$ in (2.12), we have

$$'\widetilde{R}(\overline{X},Y,Z,\overline{W}) = R(\overline{X},Y,Z,\overline{W}) + b^2[\omega(\overline{X})g(Y,\overline{W}) - \omega(Y)g(\overline{X},\overline{W})]\omega(Z)\,. \tag{2.30}$$

Subtracting (2.30) from (2.29) and using (2.11) and (1.2), we get

$$'\widetilde{R}(X,\overline{Y},\overline{Z},W) - {'\widetilde{R}}(\overline{X},Y,Z,\overline{W}) = b^2[\omega(X)g(\overline{Y},W) - \omega(\overline{Y})g(X,W)]\omega(\overline{Z}) - b^2[\omega(\overline{X})g(Y,\overline{W}) - \omega(Y)g(X,W)]\omega(Z)\,. \tag{2.31}$$

From (2.31) it is obvious that (2.18) holds if and only if

$$[\omega(X)g(\overline{Y},W) - \omega(\overline{Y})g(X,W)]\omega(\overline{Z}) = [\omega(\overline{X})g(Y,\overline{W}) - \omega(Y)g(X,W)]\omega(Z).$$

■

## 3. A special type of semi-symmetric non-metric connection

The connection $\nabla$ is said to be a special type of semi-symmetric non-metric connection if the torsion tensor $T$ and curvature tensor $\widetilde{R}$ of the connection $\nabla$ satisfy the following conditions:

$$(\nabla_X T)(Y, Z) = \omega(X)T(Y, Z)\,, \tag{3.1}$$

and

$$\widetilde{R}(X, Y)Z = 0\,. \tag{3.2}$$

Using (3.2) in (2.6), we have

$$R(X, Y)Z = b^2[\omega(Y)X - \omega(X)Y]\omega(Z)\,. \tag{3.3}$$

From (1.4), the torsion tensor T of the connection is given by

$$T(X, Y) = (a - b)[\omega(X)Y - \omega(Y)X]\,. \tag{3.4}$$

Using (3.5 in (3.4), we get

$$R(X, Y)Z = \frac{b^2}{b-a}T(X, Y)\omega(Z)\,. \tag{3.5}$$

Now, we propose:

**Theorem 3.1.** The torsion tensor of a Kähler manifold equipped with the special type of semi-symmetric non-metric connection $\nabla$, satisfies the following:

$$(i)\quad T(X, Y)\omega(\overline{Z}) = \overline{T(X, Y)}\omega(Z)\,. \tag{3.6}$$

$$(ii)\quad {}'T(X, Y, \overline{W})\omega(\overline{Z}) = {}'T(\overline{X}, \overline{Y}, W)\omega(Z)\,, \tag{3.7}$$

$$(iii)\quad {}'T(\overline{X}, \overline{Y}, \overline{W})\omega(\overline{Z}) = {}'T(X, Y, W)\omega(Z)\,, \tag{3.8}$$

$$(iv)\quad {}'T(X, Y, \overline{W})\omega(Z) + {}'T(X, Y, W)\omega(\overline{Z}) = 0\,, \tag{3.9}$$

$$(v)\quad {}'T(X, \overline{Y}, W)\omega(\overline{Z}) = {}'T(\overline{X}, Y, \overline{W})\omega(Z)\,. \tag{3.10}$$

where ${}'T(X, Y, Z) = g(T(X, Y), Z)$.

*Proof.* Using (2.7) in (3.5), we have (3.6).

Now from (3.5), we can write

$${}'R(X, Y, Z, W) = \frac{b^2}{b-a}T(X, Y, W)\omega(Z)\,. \tag{3.11}$$

Using (2.8), (2.9), (2.10), (2.11) and (3.11), we can easily get (3.7), (3.8), (3.9) and (3.10). ■

From (3.4), we have

$$(C_1^1 T)(Y) = -(a - b)(n - 1)\omega(Y)\,, \tag{3.12}$$

where $C_1^1$ denotes the operator of contraction.

Operating (3.12) by $\nabla_X$, we get

$$(\nabla_X C_1^1 T)(Y) = -(a-b)(n-1)(\nabla_X \omega)(Y)\,. \tag{3.13}$$

Contracting (3.1), we get

$$(\nabla_X C_1^1 T)(Y) = \omega(X)(C_1^1 T)(Y)\,. \tag{3.14}$$

In view of (3.12),(3.14)becomes

$$(\nabla_X C_1^1 T)(Y) = -(a-b)(n-1)\omega(X)\omega(Y)\,. \tag{3.15}$$

From (3.13) and (3.15), we have

$$(\nabla_X \omega)(Y) = \omega(X)\omega(Y)\,. \tag{3.16}$$

Now contracting (3.3), we have

$$S(Y,Z) = -b^2(n-1)\omega(Y)\omega(Z)\,. \tag{3.17}$$

Now it is known that in a Kähler manifold

$$S(\overline{Y},\overline{Z}) = S(Y,Z)\,. \tag{3.18}$$

Thus in view (3.16), (3.17) and (3.18), we conclude that

**Theorem 3.2.** In a Kähler manifold equipped with the special type of semi-symmetric non-metric connection $\nabla$, the 1-form $\omega$ and Ricci tensor $S$, satisfiy the following:

$$\begin{aligned} S(\overline{Y},\overline{Z}) &= -\,b^2(n-1)(\nabla_Y \omega)(Z) \\ &= -\,b^2(n-1)(\nabla_{\overline{Y}} \omega)(\overline{Z})\,. \end{aligned} \tag{3.19}$$

.

## 4. Second order parallel tensor

A second order tensor $\alpha$ is said to be a second order parallel tensor if $D\alpha = 0$, where $D$ denotes operator of covarient differentiation with respect to metric tensor $g$.

Putting $\omega(Y)$ in place of Y in (1.4) and using $\omega(X) = g(X,\rho)$, we get

$$(\nabla_X \omega)(Y) = (D_X \omega)(Y) - b\omega(X)\omega(Y)\,. \tag{4.1}$$

Using (3.16) in (4.1), we have

$$(D_X \omega)(Y) = (b+1)\omega(X)\omega(Y)\,. \tag{4.2}$$

Differentiating (3.4) with respect to the Riemannian connection $D$, we get

$$
\begin{aligned}
(D_X T)(Y, Z) + T(D_X Y, Z) + T(Y, D_X Z) = (a - b)[D_X \omega)(Y)(Z) \\
+ \omega(D_X Y)(Z) + \omega(Y) D_X Z - (D_X \omega)(Z)(Y) \\
- \omega(D_X Z)(Y) - \omega(Z) D_X Y] \,.
\end{aligned} \tag{4.3}
$$

And using (3.4) and (4.2), in (4.3), we obtain

$$
(D_X T)(Y, Z) = (b + 1)\omega(X) T(Y, Z) \,. \tag{4.4}
$$

Thus, we conclude

**Theorem 4.1.** In a Kähler manifold equipped with the special type of semi-symmetric non-metric connection $\nabla$, the torsion tensor will be second order parallel tensor with respect to Riemannian connection if and only if $b = -1$.

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