L (d, 2, 1)–Labeling of Helm graph

*S. Manimegala Devi

Email: s.manimegaladevi@gmail.com
Dept. of Mathematics, Sri Meenakshi Govt. Arts College for women (Autonomous)
Madurai– 625 002, Tamil Nadu, INDIA

D. S. T. Ramesh

Dept. of Mathematics, Margoschis College, Nazareth – 628 617
Tamil Nadu, INDIA.

P. Srinivasan

Dept. of Mathematics, The American college, Madurai– 625 002, Tamil Nadu, INDIA

ABSTRACT

An L(3,2,1)-labeling is a simplified model for the channel assignment problem. It is a natural generalization of the widely studied L(2,1)-labeling. The generalization of L(3,2,1)-labeling is L(d,2,1)-labeling. An L(d,2,1)-labeling of a graph G is a function f from the vertex set V(G) to the set of positive integers such that for any two vertices x, y, if d(x,y) = 1, then \(|f(x) - f(y)| \geq d\); if d(x,y) = 2, then \(|f(x) - f(y)| \geq 2\); and if d(x,y) = 3, then \(|f(x) - f(y)| \geq 1\). The L(d,2,1)-labeling number K(G) of G is the smallest positive integer k such that G has an L(d,2,1)-labeling with k as the maximum label. In this paper we determine the L(d,2,1)-labeling number of helm graph.

Keywords: L(d,2,1), helm graph.

INTRODUCTION

Griggs and Yeh defined the L(2,1)-labeling of a graph G = (V, E) as a function f which assigns every x, y in V a label from the set of positive integers such that \(|f(x) - f(y)| \geq 2\) if d(x,y) = 1 and \(|f(x) - f(y)| \geq 1\) if d(x,y) = 2 [2].
L(2,1)-labeling has been widely studied in recent years. Chartand et al. introduced the radio-labeling of graphs; this was motivated by the regulations for the channel assignments in the channel assignment problem [1]. Radio-labeling takes into consideration the diameter of the graph, and as a result, every vertex is related. Practically, interference among channels may go beyond two levels. L(3,2,1)-labeling [4] naturally extends from L(2,1)-labeling, taking into consideration vertices which are within a distance of three apart; however, it remains less difficult than radio-labeling. An L(d,2,1)-labeling [5] of a graph G = (V,E) is the generalization of L(3,2,1) labeling. [3]. In this paper we determine the L(d,2,1)-labeling number for helm graphs.

**Definition 1.1** Let G = (V,E) be a graph and f be a mapping f: V → N. f is an L(d,2,1)-labeling of G if, for all x, y in V,

\[
\left| f(x) - f(y) \right| \geq \begin{cases} 
    d, & \text{if } d(x, y) = 1 \\
    2, & \text{if } d(x, y) = 2 \\
    1, & \text{if } d(x, y) = 3
\end{cases}
\]

**Definition 1.2** The L(d,2,1)-number, \(K_d(G)\), of a graph is the smallest natural number k such that G has an L(d,2,1)-labeling with k as the maximum label. An L(d,2,1)-labeling of a graph G is called a minimal L(d,2,1)-labeling of G if, under the labeling, the highest label of any vertex is \(K_d(G)\).

**Note:** If 1 is not used as a vertex label in an L(d,2,1)-labeling of a graph, then every vertex label can be decreased by one to obtain another L(d,2,1)-labeling of the graph. Therefore in a minimal L(d,2,1)-labeling 1 will necessarily appear as a vertex label.

**Definition 1.3** A graph with the vertex set \(V = \{u_0, u_1, u_2, \ldots, u_n\}\) for \(n \geq 3\) and the edge set \(E = \{u_0u_i : 1 \leq i \leq n\} \cup \{u_1u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\}\) is called Wheel graph of length \(n\) and is denoted by \(W_n\). The vertex \(u_0\) is called the axial vertex of the wheel graph.

**Definition 1.4** The helm graph \(H_n\) is obtained from the wheel graph \(W_n\) by attaching a pendant edge at each vertex of the \(n\)-cycle of the wheel.

**Theorem 2.1:** For the helm graph \(H_n\) with all \(n \geq 4\) and \(d \geq 5\),

\[
K_d(H_n) = \begin{cases} 
    d + 2n - 1 & \text{if } n \text{ is odd; } d \leq n - 1 \\
    3d + 2 & \text{if } n \text{ is odd; } d > n - 1 \\
    a + n - 2 & \text{if } n \text{ is even and } n \geq 8 \\
    2d + n + 3 & \text{if } n \text{ is even and } n < 8
\end{cases}
\]
where \( a = \max \{2d + 3, d + n + 1\} \).

**Proof:** Let \( G = (V, E) \) be the helm graph \( H_n \) with the vertex set \( V = \{u_0, u_1, u_2, \ldots, u_n, v_1, \ldots, v_n\} \) and the edge set \( E = \{u_0u_i, u_i v_i : 1 \leq i \leq n\} \cup \{u_i u_{i+1} : 1 \leq i \leq n - 1\} \cup \{u_n u_1\} \).

We have \( d(u_0, u_i) = d(u_i, v_i) = 1 \) for all \( 1 \leq i \leq n \); \( d(u_i, u_{i+1}) = 1 \) for all \( 1 \leq i \leq n - 1 \); \( d(u_i, v_{i+1}) = 2 \) for all \( 1 \leq i \leq n - 1 \); \( d(v_i, v_j) = 4 \) for all \( 1 \leq i, j \leq n \) with \( i \neq j \), both \( i \) and \( j \) are odd or even. Therefore the \( \text{diam}(G) \) is greater than 3.

Let \( f \) be a minimal \( L(d,2,1) \)-labeling of the helm graph \( H_n \). Since the \( \text{diam}(G) \) is greater than three, the possible values of \( f(V) \) can be repeated. Since \( f \) is minimal, \( f \) takes the value 1. W.l.g, let \( f(u_0) = 1 \). Since \( d(u_0, u_i) = 1 \) for all \( 1 \leq i \leq n \), \( |f(u_0) - f(u_i)| \geq d \). Therefore \( f(u_i) \geq d + 1 \) for all \( 1 \leq i \leq n \). In particular \( f(u_1) \geq d + 1 \).

**Case A:** Let us consider the case when \( n \) is odd and \( d \leq n - 1 \).

As the distance between any two vertices of \( u_i \) with odd indices is two for \( i \geq 1 \), their labels should differ by at least two and the distance between any two vertices of \( u_i \) with even indices is two for \( i \geq 2 \), their labels should differ by at least two. As far as \( u_i \) is concerned the labeling cannot be repeated. Also, the neighboring vertices have labeling with their difference at least \( d \).

Since \( f(u_1) \geq d + 1 \) and there are \( \binom{n-1}{2} \) remaining vertices of \( u_i \) with odd indices are mutually at distance two and there are \( \binom{n-1}{2} \) remaining vertices of \( u_i \) with even indices are at distance two to each other, the minimal \( L(d, 2, 1) \)-labeling number of the helm graph \( H_n \) is greater than or equal to \( 2 \left( \binom{n-1}{2} \right) + 2 \left( \binom{n-1}{2} \right) + d + 1 \).

Hence \( K_d(H_n) \geq d + 2n - 1 \).

Next we prove that \( K_d(H_n) \leq d + 2n - 1 \).

Define

\[
    f(u_i) = \begin{cases} 
        1 & \text{if } i = 0 \\
        d + i & \text{if } i \text{ is odd; } 1 \leq i \leq n \\
        d + n + i & \text{if } i \text{ is even; } 2 \leq i \leq n - 1 
    \end{cases}
\]

\[
    f(v_i) = \begin{cases} 
        d + n + 5 & \text{if } i = 1 \\
        3 & \text{if } i \text{ is odd; } 3 \leq i \leq n \\
        4 & \text{if } i \text{ is even; } 2 \leq i \leq n - 1 
    \end{cases}
\]

As per the labeling, \( K_d(H_n) = d + 2n - 1 \) in this case. See Figure 2.2(a).
L(10, 2, 1)-labeling of the helm graph $H_{15}$.

**Figure 2.2(a)**($K_{10}(H_{15}) = 39$)

**Case B:** Let us consider the case when $n$ is odd and $d > n - 1$.
As the distance between any two of the vertices of $u_i$ with odd indices is two, their labels should differ by at least two. As far as $u_i$ is concerned the labeling cannot be repeated. Hence the minimum labels of the vertices $u_{i}$ with odd indices $u_1, u_3, \ldots, u_{n-2}$ are $d + 1, d + 3, \ldots, d + n - 2$ respectively. Since $d(u_n, u_{n-1}) = d(u_1, u_0) = 1$ and $f(u_1) \geq d + 1$, we need $f(u_n) \geq 2d + 1$ and $f(u_{n-1}) \geq 3d + 1$. Since $d(u_{n-1}, v_1) = 3$, the minimal $L(d, 2, 1)$-labeling number of $H_n$ is greater than or equal to $3d + 2$.
Hence $K_d(H_n) \geq 3d + 2$.
Next we prove that $K_d(H_n) \leq 3d + 2$.
Define

$$f(u_i) = \begin{cases} 1 & \text{if } i = 0 \\ d + i & \text{if } i \text{ is odd; } 1 \leq i \leq n - 2 \\ 2d + i + 1 & \text{if } i \text{ is even; } 2 \leq i \leq n - 3 \\ 3d + 1 & \text{if } i = n - 1 \\ 2d + 1 & \text{if } i = n \end{cases}$$
$f(v_i) = \begin{cases} 
3d + 2 & \text{if } i = 1 \\
3 & \text{if } i \text{ is odd; } 3 \leq i \leq n \\
4 & \text{if } i \text{ is even; } 2 \leq i \leq n - 1 
\end{cases}$

As per the labeling $K_d(H_n) = 3d + 2$ in this case. See Figure 2.2(b).

$L(14, 2, 1)$-labeling of the helm graph $H_{11}$.

**Figure 2.2(b)($K_{14}(H_{11}) = 44$)**

**Case C:** Let us consider the case when $n$ is even and $n \geq 8$.

As the distance between any two of the vertices of $u_i$ with odd indices is two, their labels should differ by at least two. As far as $u_i$ is concerned, the labelling cannot be repeated. Also, the neighboring vertices have labeling with their difference at least $d$. Hence the minimum labels of the vertices $u_1, u_3, \ldots, u_{n-1}$ are $d + 1, d + 3, \ldots, d + n - 1$ respectively. Since $d(u_{n-1}, u_2) = 2$, $d(u_2, u_3) = 1$, $f(u_3) \geq d + 3$ and $f(u_{n-1}) \geq d + n - 1$, the minimum label for $u_2$ is $\max \{2d + 3, d + n + 1\}$. Let $a = \max \{2d + 3, d + n + 1\}$. Therefore $f(u_2) \geq a$.

As the distance between any two of the vertices of $u_i$ with even indices is two, their labels should differ by at least two. Since $f(u_2) \geq a$ and there are $\left\lfloor \frac{n}{2} \right\rfloor - 1$ remaining vertices of $u_i$ with even indices are at distance two to each other, the minimal $L(d, 2, 1)$-labeling number of the helm graph $H_{n\text{is}}$ greater than or equal to $a + 2\left(\frac{n}{2} - 1\right)$.

Hence $K_d(H_n) \geq a + n - 2$. 
Next we prove that $K_d(H_n) \leq a + n - 2$.

Define

$$f(u_i) = \begin{cases} 
1 & \text{if } i = 0 \\
d + i & \text{if } i \text{ is odd; } 1 \leq i \leq n - 1 \\
a + i - 2 & \text{if } i \text{ is even; } 2 \leq i \leq n 
\end{cases}$$

$$f(v_i) = \begin{cases} 
a + 3 & \text{if } i = 1 \\
3 & \text{if } i \text{ is odd; } 3 \leq i \leq n - 1 \\
4 & \text{if } i \text{ is even; } 2 \leq i \leq n 
\end{cases}$$

As per the labelling $K_d(H_n) = a + n - 2$ where $a = \max\{2d + 3, d + n + 1\}$ in this case. See Figure 2.2(c).

L$(18, 2, 1)$-labeling of the helm graph $H_{16}$.

**Figure 2.2(c)**($K_{18}(H_{16}) = 53$)

**Case D:** Let us consider the case when $n$ is even and $n < 8$.

As the distance between any two of the vertices of $u_i$ with odd indices is two, their labels should differ by at least two. As far as $u_i$ is concerned the labelling cannot be repeated.

Also, the neighboring vertices have labeling and at least $d$. 

![Diagram of helm graph $H_{16}$ labeled with $K_{18}$]
Hence the minimum labels of the vertices $u_1, u_3, \ldots, u_{n-1}$ are $d + 1$, $d + 3, \ldots$, $d + n - 1$ respectively. Since $d(u_{n-1}, u_2) = 2$, $d(u_2, u_3) = 1$, $f(u_3) \geq d + 3$ and $f(u_{n-1}) \geq d + n - 1$, the minimum label for $u_2$ is $2d + 3$. As the distance between any two of the vertices of $u_i$ with even indices is two, their labels should differ by at least two. Since $f(u_2) \geq 2d + 3$ and there are $\binom{n}{2} - 1$ remaining vertices of $u_i$ with even indices are at distance two to each other and $d(u_n, v_1) = 2$, the minimal $L(d, 2, 1)$-labeling number of the helm graph $H_n$ is greater than or equal to $2d + 3 + 2\left(\frac{n}{2} - 1\right) + 2$.

Next we prove that $K_d(H_n) \leq 2d + n + 3$.

Define

$$f(u_i) = \begin{cases} 
1 & \text{if } i = 0 \\
\begin{align*}
d + i & \quad \text{if } i \text{ is odd; } 1 \leq i \leq n - 1 \\
2d + i + 1 & \quad \text{if } i \text{ is even; } 2 \leq i \leq n
\end{align*}
\end{cases}$$

$$f(v_i) = \begin{cases} 
2d + n + 3 & \text{if } i = 1 \\
\begin{align*}
3 & \quad \text{if } i \text{ is odd; } 3 \leq i \leq n - 1 \\
4 & \quad \text{if } i \text{ is even; } 2 \leq i \leq n
\end{align*}
\end{cases}$$

As per the labeling $K_d(H_n) = 2d + n + 3$ in this case. See Figure 2.2(d).

$L(9, 2, 1)$-labeling of the helm graph $H_6$.

![Figure 2.2(d)(K₉(H₆) =27).](image-url)
Remark 2.3: When \( d = 4 \), we get the same result as in the above theorem by assigning a minimum label 10 to \( v_2 \).

References: