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L (d, 2, 1)–Labeling of Helm graph

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ABSTRACT

An L(3,2,1)-labeling is a simplified model for the channel assignment problem.
It is a natural generalization of the widely studied L(2,1)-labeling. The generalization of L(3,2,1)-labeling is L(d,2,1)-labeling. An L(d,2,1)-labeling of a graph G is a function f from the vertex set V(G) to the set of positive integers such that for any two vertices x , y , if d(x,y) =1 , then $|f(x)-f(y)| \geq d$; if d(x,y) = 2 ,then $|f(x)-f(y)| \geq 2$; and if d(x,y) = 3 , then $|f(x)-f(y)| \geq 1$.The L(d,2,1)-labeling number K(G) of G is the smallest positive integer k such that G has an L(d,2,1)-labeling with k as the maximum label. In this paper we determine the L(d,2,1)-labeling number of helm graph.

Keywords : L(d,2,1), helm graph.

INTRODUCTION

Griggs and Yeh defined the L(2,1)-labeling of a graph G = (V, E) as a function f which assigns every x , y in V a label from the set of positive integers such that $|f(x)-f(y)| \geq 2$ if d(x,y) = 1 and $|f(x)-f(y)| \geq 1$ if d(x,y) = 2 [2].

L(2,1)-labeling has been widely studied in recent years.
Chartand et al. introduced the radio - labeling ofgraphs; this was motivated by the regulations for the channel assignments in the channel assignment problem[1].Radio – labeling takes into consideration the diameter of the graph, and as a result, every vertex is related.
Practically , interference among channels may go beyond two levels. L(3,2,1)-labeling [4] naturally extends from L(2,1)-labeling, taking into consideration vertices which are within a distance of three apart ; however, it remains less difficult than radio – labeling. An L(d,2,1)-labeling [5] of a graph G = (V,E) is the generalization of L(3,2,1)labeling. [3].
In this paper we determine the L(d,2,1)-labeling number for helm graphs.

Definition 1.1 Let G = (V,E) be a graph and f be a mapping f: V → N. f is an L(d,2,1)-labeling of G if, for all x, y in V,

$$|f(x) - f(y)| \geq \begin{cases} d, & \text{if } d(x, y) = 1 \\ 2, & \text{if } d(x, y) = 2 \\ 1, & \text{if } d(x, y) = 3 \end{cases}$$

Definition 1.2 The L(d,2,1)-number, $K_d(G)$, of a graph is the smallest natural number k such that G has an L(d,2,1)-labeling with k as the maximum label. An L(d,2,1)-labeling of a graph G is called a minimal L(d,2,1)-labeling of G if, under the labeling, the highest label of any vertex is $K_d(G)$.

Note: If 1 is not used as a vertex label in an L(d,2,1)-labeling of a graph, then every vertex label can be decreased by one to obtain another L(d,2,1)-labeling of the graph.Therefore in a minimal L(d,2,1)-labeling 1 will necessarily appear as a vertex label.

Definition 1.3 A graph with the vertex set $V = \{u_0,u_1,u_2,\ldots,u_n\}$ for $n \geq 3$ and the edge set $E = \{u_0u_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nu_1\}$ is called ***Wheel graph*** of length n and is denoted by $\mathbf{W_n}$. The vertex u_0 is called the axial vertex of the wheel graph.

Definition 1.4 The helm graph H_n is obtained from the wheel graph W_n by attaching a pendent edge at each vertex of the n-cycle of the wheel.

Theorem 2.1: For the helm graph H_n with all $n \geq 4$ and $d \geq 5$,

$$K_d(H_n) = \begin{cases} d + 2n - 1 & \text{if } n \text{ is odd; } d \leq n - 1 \\ 3d + 2 & \text{if } n \text{ is odd; } d > n - 1 \\ a + n - 2 & \text{if } n \text{ is even and } n \geq 8 \\ 2d + n + 3 & \text{if } n \text{ is even and } n < 8 \end{cases}$$

where $a = \max\{2d + 3, d + n + 1\}$.

Proof : Let $G = (V, E)$ be the helm graph H_n with the vertex set $V = \{u_0, u_1, u_2,\ldots, u_n, v_1, ,\ldots, v_n\}$ and the edge set $E = \{u_0u_i, u_iv_i: 1 \le i \le n\} \cup \{u_iu_{i+1} : 1 \le i \le n-1\} \cup \{u_nu_1\}$.
We have $d(u_0, u_i) = d(u_i, v_i) = 1$ for all $1 \le i \le n$; $d(u_i, u_{i+1}) = 1$ for all $1 \le i \le n-1$; $d(u_i, v_{i+1}) = 2$ for all $1 \le i \le n-1$; $d(v_i, v_j) = 4$ for all $1 \le i, j \le n$ with $i \ne j$, both i and j are odd or even. Therefore the diam(G) is greater than 3.
Let f be a minimal L(d,2,1)-labeling of the helm graph H_n. Since the diam(G) is greater than three, the possible values of f(V) can be repeated.
Since f is minimal, f takes the value 1. W.l.g, let $f(u_0) = 1$. Since $d(u_0, u_i) = 1$ for all $1 \le i \le n$, $|f(u_0) - f(u_i)| \ge d$. Therefore $f(u_i) \ge d + 1$ for all $1 \le i \le n$. In particular $f(u_1) \ge d + 1$.

Case A: Let us consider the case when n is odd and $d \le n - 1$.
As the distance between any two vertices of u_i with odd indices is two for $i \ge 1$, their labels should differ by atleast two and the distance between any two vertices of u_i with even indices is two for $i \ge 2$, their labels should differ by atleast two.As far as u_i is concerned the labeling cannot be repeated. Also,the neighboring vertices have labeling with their difference atleast d.
Since $f(u_1) \ge d + 1$ and there are $\left(\frac{n-1}{2}\right)$ remaining vertices of u_i with odd indices are mutually at distance two and there are $\left(\frac{n-1}{2}\right)$ remaining vertices of u_i with even indices are at distance two to each other, the minimal L(d, 2, 1)-labeling number of the helm graph H_n is greater than or equal to $2\left(\frac{n-1}{2}\right) + 2\left(\frac{n-1}{2}\right) + d + 1$.
Hence $K_d(H_n) \ge d + 2n - 1$.
Next we prove that $K_d(H_n) \le d + 2n - 1$.

Define

$$f(u_i) = \begin{cases} 1 & \text{if } i = 0 \\ d + i & \text{if } i \text{ is odd; } 1 \le i \le n \\ d + n + i & \text{if } i \text{ is even; } 2 \le i \le n - 1 \end{cases}$$

$$f(v_i) = \begin{cases} d + n + 5 & \text{if } i = 1 \\ 3 & \text{if } i \text{ is odd; } 3 \le i \le n \\ 4 & \text{if } i \text{ is even; } 2 \le i \le n - 1 \end{cases}$$

As per the labeling, $K_d(H_n) = d + 2n - 1$in this case. See Figure 2.2(a)

L(10, 2, 1)-labeling of the helm graph H_{15}.

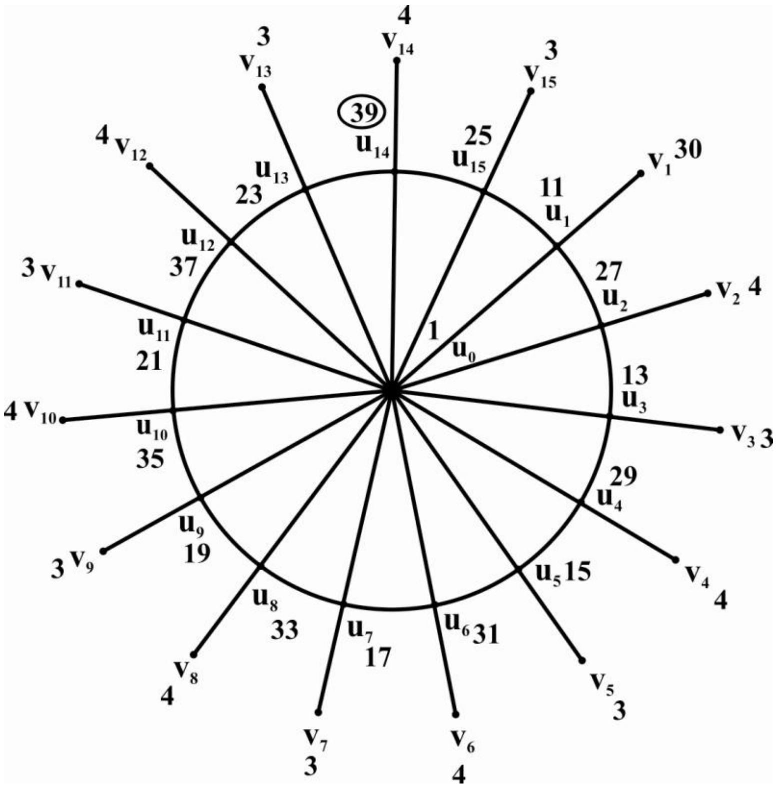

Figure 2.2(a)($K_{10}(H_{15}) = 39$)

Case B: Let us consider the case when n is odd and $d > n - 1$.
As the distance between any two of the vertices of u_i with odd indices is two, their labels should differ by atleast two. As far as u_i is concerned the labeling cannot be repeated. Hence the minimum labels of the vertices u_i with odd indices $u_1, u_3,\ldots,u_{n-2}$ are $d + 1, d + 3,\ldots, d + n - 2$ respectively. Since $d(u_n, u_{n-1}) = d(u_1, u_n) = 1$ and $f(u_1) \geq d + 1$, we need $f(u_n) \geq 2d + 1$ and $f(u_{n-1}) \geq 3d + 1$. Since $d(u_{n-1}, v_1) = 3$, the minimal L(d, 2, 1)-labeling number of H_n is greater than or equal to $3d + 2$.
Hence $K_d(H_n) \geq 3d + 2$.
Next we prove that $K_d(H_n) \leq 3d + 2$.
Define

$$f(u_i) = \begin{cases} 1 & \text{if } i = 0 \\ d + i & \text{if } i \text{ is odd; } 1 \leq i \leq n - 2 \\ 2d + i + 1 & \text{if } i \text{ is even; } 2 \leq i \leq n - 3 \\ 3d + 1 & \text{if } i = n - 1 \\ 2d + 1 & \text{if } i = n \end{cases}$$

$$f(v_i) = \begin{cases} 3d + 2 & \text{if } i = 1 \\ 3 & \text{if } i \text{ is odd; } 3 \le i \le n \\ 4 & \text{if } i \text{ is even; } 2 \le i \le n-1 \end{cases}$$

As per the labeling $K_d(H_n) = 3d + 2$ in this case. See Figure 2.2(b).
L(14, 2, 1)-labeling of the helm graph H_{11}.

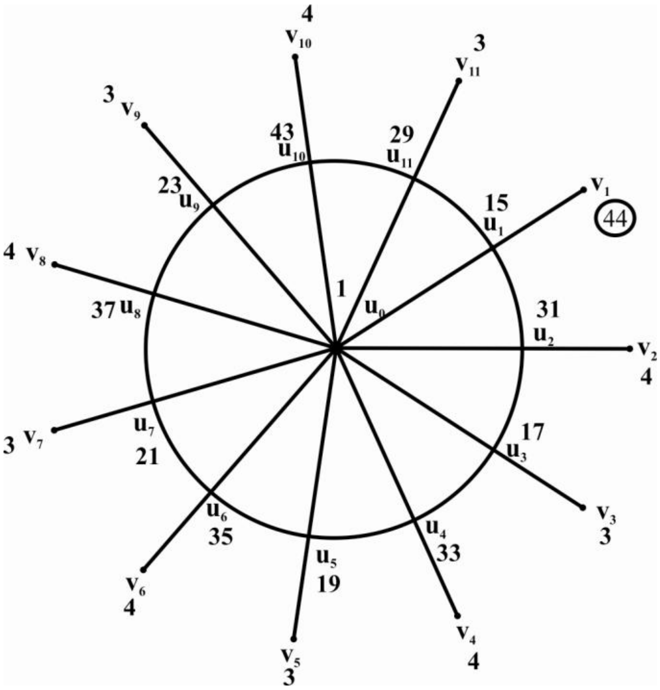

Figure 2.2(b)($K_{14}(H_{11})$= 44)

Case C: Let us consider the case when n is even and $n \ge 8$.
As the distance between any two of the vertices of u_i with odd indices is two, their labels should differ by atleast two. As far as u_i is concerned the labelling cannot be repeated. Also, the neighboring vertices have labeling with their difference atleast d. Hence the minimum labels of the vertices $u_1, u_3, \ldots, u_{n-1}$ are $d + 1, d + 3, \ldots, d + n - 1$ respectively. Since $d(u_{n-1}, u_2) = 2$, $d(u_2, u_3) = 1$, $f(u_3) \ge d + 3$ and $f(u_{n-1}) \ge d + n - 1$, the minimum label for u_2 is max $\{2d + 3, d + n + 1\}$. Let a = max $\{2d + 3, d + n + 1\}$. Therefore $f(u_2) \ge a$.
As the distance between any two of the vertices of u_i with even indices is two, their labels should differ by atleast two. Since $f(u_2) \ge a$ and there are $\left(\frac{n}{2} - 1\right)$ remaining vertices of u_i with even indices are at distance two to each other, the minimal L(d, 2, 1)-labeling number of the helm graph H_n is greater than or equal to $a + 2\left(\frac{n}{2} - 1\right)$.
Hence $K_d(H_n) \ge a + n - 2$.

Next we prove that $K_d(H_n) \leq a + n - 2$.

Define

$$f(u_i) = \begin{cases} 1 & \text{if } i = 0 \\ d + i & \text{if } i \text{ is odd; } 1 \leq i \leq n-1 \\ a + i - 2 & \text{if } i \text{ is even; } 2 \leq i \leq n \end{cases}$$

$$f(v_i) = \begin{cases} a + 3 & \text{if } i = 1 \\ 3 & \text{if } i \text{ is odd; } 3 \leq i \leq n-1 \\ 4 & \text{if } i \text{ is even; } 2 \leq i \leq n \end{cases}$$

As per the labelling $K_d(H_n) = a + n - 2$ where $a = \max\{2d + 3, d + n + 1\}$ in this case. See Figure2.2(c).
L(18, 2, 1)-labeling of the helm graph H_{16}.

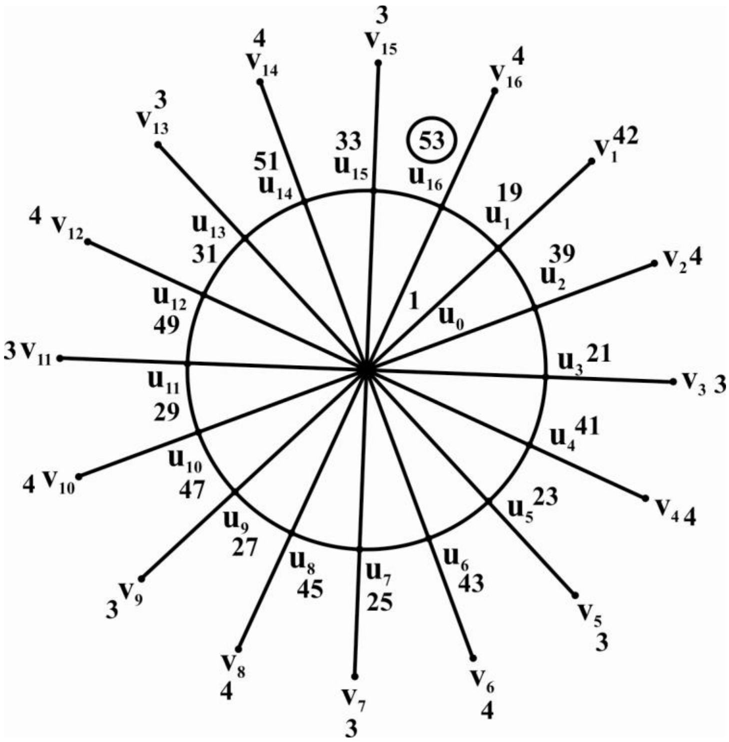

Figure 2.2(c)($K_{18}(H_{16}) = 53$)

Case D: Let us consider the case when n is even and $n < 8$.
As the distance between any two of the vertices of u_i with odd indices is two, their labels should differ by atleast two. As far as u_i is concerned the labelling cannot be repeated.
Also, the neighboring vertices have labeling and atleast d.

Hence the minimum labels of the vertices $u_1, u_3, \ldots, u_{n-1}$ are $d + 1, d + 3, \ldots, d + n - 1$ respectively. Since $d(u_{n-1}, u_2) = 2$, $d(u_2, u_3) = 1$, $f(u_3) \geq d + 3$ and $f(u_{n-1}) \geq d + n - 1$, the minimum label for u_2 is $2d + 3$.
As the distance between any two of the vertices of u_i with even indices is two, their labels should differ by atleast two. Since $f(u_2) \geq 2d + 3$ and there are $\left(\frac{n}{2} - 1\right)$ remaining vertices of u_i with even indices are at distance two to each other and $d(u_n, v_1) = 2$, the minimal L(d, 2, 1)-labeling number of the helm graph H_n is greater than or equal to $2d + 3 + 2\left(\frac{n}{2} - 1\right) + 2$.
Next we prove that $K_d(H_n) \leq 2d + n + 3$.

Define

$$f(u_i) = \begin{cases} 1 & \text{if } i = 0 \\ d + i & \text{if } i \text{ is odd; } 1 \leq i \leq n - 1 \\ 2d + i + 1 & \text{if } i \text{ is even; } 2 \leq i \leq n \end{cases}$$

$$f(v_i) = \begin{cases} 2d + n + 3 & \text{if } i = 1 \\ 3 & \text{if } i \text{ is odd; } 3 \leq i \leq n - 1 \\ 4 & \text{if } i \text{ is even; } 2 \leq i \leq n \end{cases}$$

As per the labeling $K_d(H_n) = 2d + n + 3$ in this case. See Figure 2.2(d).
L(9, 2, 1)-labeling of the helm graph H_6.

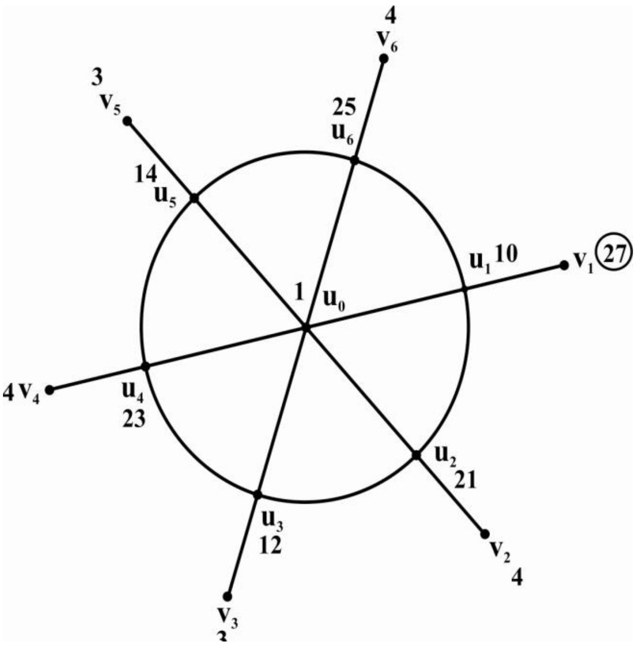

Figure 2.2(d)($K_9(H_6)$ =27).

Remark2.3: When d = 4,we get the same result as in the above theorem by assigning a minimum label 10 to v_2.

References:

[1] G. Chartrand, D. Erwin, F. Harary, and P. Zhang, Radio labeling of graphs, Bull . Inst . combin . Appl., 33(2001), 77-85.

[2] J.R.Griggs and R.K.Yeh, Labeling graphs with a condition at distance two, SIAM J. Discrete Math.,5(1992) 586– 5995.

[3] L.Jia-Zhuang and S.Zhen-dong, TheL(3,2,1)-labeling problem on graphs, Mathematical application, 17 (4) (2004), 596 – 602.

[4] D.S.T.RAMESH and Sujatha Sarathi, L(3,2,1)-labeling of some special graphs, Archimedes J. Math., 2(1)(2012), 31-38.

[5] D. S. T. RAMESH and Sujatha Sarathi, L(d, 2, 1)-labeling of some tree graphs, Archimedes J. Math., 3(1)(2013), 59-65.