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Stability Theorems of Impulsive Infinite Delay Differential Equations

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Abstract

In this paper, the stability of impulsive infinite delay differential equations has been considered. By using Lyapunov functions and Razumikhin techniques, some criteria of uniform stability and asymptotic stability are provided. We adopt several Lyapunov functionals so that not only can they be easier constructed, but also the conditions ensuring the required stability are less restrictive.

Keywords: Impulsive infinite delay differential equations; Uniform asymptotic stability; Razumikhin techniques; Lyapunov functionals.

Subject classification codes: 34K20; 93D05; 34K38

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1. INTRODUCTION

It is known that many biological phenomenon involving thresholds, bursting rhythm models in medicine and biology optimal control models in economics and frequency modulate system exhibit the impulse effect. Thus impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena for several real world problems. In recent years, qualitative properties of the mathematical theory of impulsive differential equations have been developed by large number of mathematicians ; see [1–12] . Systems with infinite delay deserve study because they describe a kind of system present in the real world. In [4], Lyapunov functionals are adopted and components of x are divided into several groups, correspondingly, several functions $V_j\left(t, x^{(j)}\right), (j = 1,2,\ldots\ldots, m)$ are employed. In that way, to construct the suitable function is rather easy and the imposed conditions ensuring the required stability are less restrictive. There are some results on systems with infinite delay see [13, 14]

In this work, we consider impulsive infinite delay differential equations. By using Lyapunov function and the Razumikhin technique ; we get some results that are more general than the ones given in [5]. We extend the new technique developed in [4] to study impulsive systems. We give an example to show that this new technique is rather effective and especially applicable to system of impulsive infinite delay differential equations.

2. PRELIMINARIES

Consider the following, impulsive infinite delay differential equations

$$x'(t) = f\left(t, x(t), x\big(t - \tau(t)\big)\right), \qquad t \geq t_0, \qquad t \neq t_k$$

$$\Delta x(t) = x(t) - x(t-) = I_k(x(t-)), \quad t = t_k;\ k = 1,2,\ldots\ldots. \tag{2.1}$$

Where $t \in R^+$, $f \in C[R^+ \times R^n \times PC\big((-\infty, 0], R^n\big), R^n]$, $PC((-\infty, 0], R^n)$ denotes the space of piecewise right continuous functions $\emptyset: (-\infty, 0] \to R^n$ with the sup norm $\|\emptyset\| = \sup_{-\infty < s \leq 0} |\emptyset(s)|$, $|.|$ is a norm in R^n, $f(t, 0,0) \equiv 0, I_k(0) \equiv 0$, $t \geq \tau(t) \geq 0$, $0 = \tau_0 < \tau_1 < \tau_2 < \cdots \ldots \ldots < \tau_k < \cdots \ldots$, $\tau_k \to \infty$ for $k \to \infty$, $x(t^+) = \lim_{s \to t^+} x(s)$, and $x(t^-) = \lim_{s \to t^-} x(s)$. The functions $I_k: R^n \to R^n$, $k = 1,2,\ldots\ldots$, and such that if $\|x\| < H$ and $I_k(x) \neq 0$, then $\|x + I_k(x)\| < H$, where $H =$ const.> 0. The initial condition for system (2.1) is given by

$$x_\sigma = \emptyset \tag{2.2}$$

Where $\emptyset \in PC((-\infty, 0], R^n)$

We assume that a solution for the initial value problem (2.1) and (2.2) does exist and is unique. Since $f(t,0,0) = 0$, then $x(t) = 0$ is a solution of (2.1), which is called zero solution.

Let $$PC(\rho) = \{\emptyset \in PC((-\infty, 0], R^n) : \|\emptyset\| < \rho\}$$

For $\emptyset \in PC(\rho)$, we define

$$\|\emptyset\| = \|\emptyset\|^{(-\infty,t]} = \sup_{-\infty \le s \le t} |\emptyset(s)|$$

For convenience, we define $|x| = \max_{1 \le i \le n} |x_i|$, for $x \in R^n$.

We introduce some definitions as follows:

Definition 2.1 The zero solution of (2.1) and (2.2) is said to be stable if for any $\sigma \ge t_0$ and $\epsilon > 0$, there is a $\delta = \delta(\sigma, \epsilon)$ such that $[\emptyset \in PC(\delta), t \ge \sigma]$ implies that $|x(t, \sigma, \emptyset)| \le \epsilon$.

Definition 2.2 The zero solution of (2.1) and (2.2) is said to be uniformly stable if it is stable and δ is independant of σ.

Definition 2.3 A continuous $W: R^+ \to R^+$ is called a wedge function if $W(0) = 0$ and $W(s)$ is strictly increasing.

The following lemma (c.f [1]) is needed in proving the main result.

Lemma: Let u be a continuous and bounded function. Then for any wedge functions W and W^*, any $h > 0$, and for each $\beta > 0$, there is a corresponding $\beta^* > 0$ such that

$$\int_{t-h}^{t} W(|u(s)|)ds \ge \beta \quad \text{implies} \quad \int_{t-h}^{t} W^*(|u(s)|)ds \ge \beta^*$$

In what follows, we will split $\emptyset = (\emptyset_1, \emptyset_2, \emptyset_3, \dots \dots \dots., \emptyset_n)^T \in PC$ into several vectors, say,

$$(\emptyset_1{}^{(1)}, \emptyset_2{}^{(1)}, \dots \dots \dots.., \emptyset_{n_1}{}^{(1)})^T, (\emptyset_1{}^{(2)}, \emptyset_2{}^{(2)}, \dots \dots \dots.., \emptyset_{n_2}{}^{(2)})^T, .., (\emptyset_1{}^{(m)}, \emptyset_2{}^{(m)}, \dots \dots \dots.., \emptyset_{n_m}{}^{(m)})^T$$

such that $n_1 + n_2 + \cdots \dots \dots. + n_m = n$ and $\{\emptyset_1{}^{(1)}, .., \emptyset_{n_1}{}^{(1)},$
$\emptyset_1{}^{(2)}, ..., \emptyset_{n_2}{}^{(2)}, \emptyset_1{}^{(m)}, .., \emptyset_{n_m}{}^{(m)}\} = \{\emptyset_1, \emptyset_2, \dots \dots..\}$

For convenience, we define

$$\emptyset^{(j)} = \left(\emptyset_1{}^{(j)}, \emptyset_2{}^{(j)}, \dots \dots \dots .., \emptyset_{n_j}{}^{(j)}\right), \quad j = 1,2, \dots \dots \dots, m$$

And $$\emptyset = \left(\emptyset^{(1)}, \emptyset^{(2)}, \dots \dots \dots ., \emptyset^{(m)}\right)^T$$

Note that the order of components in $\emptyset^{(j)}$ is not necessarily same as that in $\emptyset$

For $x = (x_1, x_2, \dots \dots \dots \dots, x_n)^T \in R^n$, we adopt the similar notation as for $\emptyset \in PC(\rho)$

Let $$\left|\emptyset^{(j)}\right| = \max_{1 \le k \le n_j} \left|x_k{}^{(j)}\right|, \; j = 1,2, \dots \dots \dots \dots, m$$

and thus

$$|x| = \max_{1 \le j \le m} \left|x^{(j)}\right|$$

Correspondingly

$$\left|\emptyset^{(j)}(s)\right| = \max_{1 \le k \le n_j} \left|x_k{}^{(j)}(s)\right|, \; j = 1,2, \dots \dots \dots \dots, m \quad \text{and} \quad |\emptyset(s)| = \max_{1 \le j \le m} \left|\emptyset^{(j)}(s)\right|$$

Let

$$\left\|\emptyset^{(j)}\right\| = \left\|\emptyset^{(j)}\right\|^{(-\infty, t]} = \sup_{-\infty \le s \le t} \left|\emptyset^{(j)}(s)\right|, \quad j = 1,2, \dots \dots, m$$

and denote

$$PC^{(j)}(t) = \left\{\emptyset^{(j)}: (-, t] \to R^{n_j} \mid \emptyset^{(j)} \text{ is continuous and bounded}\right\},$$

and

$$PC_\rho{}^{(j)}(t) = \left\{\emptyset^{(j)} \in PC^{(j)}(t) \mid \left\|\emptyset^{(j)}\right\| < \rho \right\}$$

3. MAIN RESULTS

Theorem 3.1: let $\Phi_j: R^+ \to R^+$ be continuous, $\Phi_j \in L^1[0, \infty), \Phi_j(t) \le K_j$ for $t \ge 0$ with some constants $K_j (j = 1,2)$ and $W_{ij} (i = 1,2,3,4,5, j = 1,2)$ be wedge functions. Let there exist continuous functions $P_j, q_j: (0, \infty) \to (0, \infty)$ such that $P(s) > s$ for $s > 0$ and $q_j(s)$ is non increasing $(j = 1,2)$ and continuous lyapunov functionals $V_j: [0, \infty) \times PC_H{}^{(j)}(t) \to R^+ (j = 1,2)$

i) $$W_{1j}(\left|\emptyset^{(j)}(t)\right| \le V_j(t, \emptyset^{(j)}(t)) \le W_{2j}(\left|\emptyset^{(j)}(t)\right| + W_{3j}\left[\int_{-\infty}^{t} \Phi_j(t-s) W_{4j}\left|\emptyset^{(j)}(s)\right| ds\right] \; j = 1,2$$

ii) When $V_1(t) \geq V_2(t)$ there holds $V_1'(t) \leq -W_{51}(|x^{(1)}(t)|$ if $V_1(t-\tau(t)) < P_1(V_1(t))$

When $V_1(t) \leq V_2(t)$ there holds $V_2'(t) \leq -W_{52}(|x^{(2)}(t)|$ if $V_2(t-\tau(t)) < P_2(V_2(t))$

iii) $V_j\left(\tau_k x(\tau_k{}^-) + I_k\left(x(\tau_k{}^-)\right)\right) \leq (1+b_k)V_j(\tau_k{}^-, x(\tau_k{}^-)),\ j = 1,2, k = 1,2, \dots \dots$ for which $b_k \geq 0$ and $\sum_{k=1}^{\infty} b_k < \infty$

Where $x(t) = (x^{(1)}(t), x^{(2)}(t))$ is a solution of (2.1) and (2.2) then the zero solution of (2.1) is U.A.S

Proof: Since $b_k \geq 0$ and $\sum_{k=1}^{\infty} b_k < \infty$, it follows that $\prod_{k=1}^{\infty}(1+b_k) = M$ and $1 \leq M \leq \infty$

Let $x(t) = \{x^{(1)}(t), x^{(2)}(t)\}$ is a solution of (2.1). Define a function $V(t)$ as follows:

$$V(t) = \begin{Bmatrix} V_1(t), & if & V_1(t) \geq V_2(t) \\ V_1(t), & if & V_1(t) \leq V_2(t) \end{Bmatrix} \tag{3.1}$$

Then $V(t)$ is continuous for all $t \in R^+$

We first claim that for any $t \in R^+$,

$$\frac{[W_{11}(|x^{(1)}(t)|) + W_{12}(|x^{(2)}(t)|)]}{2} \leq V(t)$$

$$\leq W_{21}(|x^{(1)}(t)|) + W_{22}(|x^{(2)}(t)|) + W_{31}\left[\int_{-\infty}^{t} \Phi_1(t-s)W_{41}(|x^{(1)}(s)|)ds\right]$$

$$+ W_{32}\left[\int_{-\infty}^{t} \Phi_2(t-s)W_{42}(|x^{(2)}(s)|)ds\right] \tag{3.2}$$

In fact if $V_1(t) \geq V_2(t)$ then by condition (i),

$$V(t) = V_1(t) \geq \frac{[V_1(t) + V_2(t)]}{2} \geq \frac{[W_{11}(|x^{(1)}(t)|) + W_{12}(|x^{(2)}(t)|)]}{2}$$

Whereas if $V_1(t) \leq V_2(t)$, we also have

$$V(t) = V_2(t) \geq \frac{[V_1(t) + V_2(t)]}{2} \geq \frac{[W_{11}(|x^{(1)}(t)|) + W_{12}(|x^{(2)}(t)|)]}{2}$$

On the other hand, the right hand inequality in (3.2) obviously holds.

Let $P(s) = \min\{P_1(s), P_2(s)\}$ and $q(s) = max\{q_1(s), q_2(s)\}$

Obviously P and $q: (0, \infty) \to (0, \infty)$ are also continuous, $P(s) > s$ for $s > 0$ and $q(s)$ is non increasing.

Now we can show that

(a) On any subinterval of $[t_0, \infty)]$ where $V_1(t) \geq V_2(t)$, we have

$$V'(t) \leq -W_{51}(\left|x^{(1)}(t)\right| \tag{3.3}$$

if $$V\big(t - \tau(t)\big) < P\big(V(t)\big)$$

(b) On any subinterval of $[t_0, \infty)]$ where $V_1(t) \leq V_2(t)$, we have

$$V'(t) \leq -W_{52}(\left|x^{(2)}(t)\right| \tag{3.4}$$

if $$V\big(t - \tau(t)\big) < P\big(V(t)\big)$$

In fact, suppose that there is some $s_1 > s_0$ such that $V(t) \geq V_2(t)$ for $t \in [s_0, s_1]$

Then by (3.1), we have

$$V(t) = V_1(t), \text{ for } t \in [s_0, s_1]$$

If $V_1(t - \tau(t)) \geq V_2(t - \tau(t))$ then $V(t - \tau(t)) = V_1(t - \tau(t))$ and $P\big(V(t)\big) = P(V_1(t)) \leq P_1(V_1(t))$, hence $V(t - \tau(t)) < P(V(t))$ implies $V_1\big(t - \tau(t)\big) < P_1(V_1(t))$; whereas if $V_1(t - \tau(t)) \leq V_2(t - \tau(t))$ then $V_1\big(t - \tau(t)\big) \leq V_2\big(t - \tau(t)\big) = V(t - \tau(t))$ and $P\big(V(t)\big) = P(V_1(t)) \leq P_1(V_1(t))$, hence $V(t - \tau(t)) < P(V(t))$ also implies $V_1\big(t - \tau(t)\big) < P_1(V_1(t))$.

We conclude that for any $t \in [s_0, s_1]$,

$$V'(t) = V_1'(t) \leq -W_{51}\big(\left|x^{(1)}(t)\right|\big) \text{ if } V\big(t - \tau(t)\big) < P(V(t))$$

In a similar way

$$V'(t) = V_2'(t) \leq -W_{52}\big(\left|x^{(2)}(t)\right|\big) \text{ if } V\big(t - \tau(t)\big) < P(V(t))$$

The trivial solution of (2.1) is trivially U.S

Furthermore, we can show the U.A.S. Let $\epsilon = h < H$, we can find the corresponding $\delta(h) > 0$ $(\delta < h)$ in the U.S, and let $\eta = \delta(h)$ then $[\sigma \geq t_0, \emptyset \in PC_\eta(\delta), t \geq \alpha]$ implies

$$V(t) \leq \frac{h^*}{2} \text{ and } |x(t)| \leq h \tag{3.5}$$

Where $Mh^* = min\{W_{11}(h), W_{12}(h)\}$

For any given $\gamma > 0$ with $\gamma < h$, we will find a $T(\gamma) > 0$ such that $[\sigma \geq t_0, \emptyset \in PC(\delta), t \geq \sigma + T]$ imply $|x(t)| = |x(t, \sigma, \emptyset| \leq \gamma$. Since $\emptyset_j \in L^1[0, \infty)$, for the given $\gamma > 0$, let $M\gamma^* = \min\{W_{11}(\gamma), W_{12}(\gamma)\}$, we can find $l > 1$ such that

$$W_{4j}(h) \int_l^\infty \Phi_j(s) ds < W_{3j}{}^{-1} \frac{(\frac{\gamma^*}{4})}{2}$$

And

$$W_{2j} \left\{ W_{4j}{}^{-1} \left[W_{3j}{}^{-1} \frac{(\frac{M\gamma^*}{4})}{(2K_j l} \right] \right\} < \frac{\gamma^*}{4}\ ,\ j = 1,2..$$

Then for $t \geq \sigma + l$, we have by (i) that

$$V_j(t) \leq W_{2j}\left(\left|x^{(j)}(t)\right|\right) + W_{3j}\left[W_{3j}{}^{-1} \frac{(\frac{\gamma^*}{4})}{2} + K_j \int_{t-l}^t W_{4j}\left(\left|x^{(j)}(s)\right|\right) ds\right], j = 1,2.. \quad (3.6)$$

Let $0 < a < \inf\{P(s) - s\left(\frac{\gamma^*}{2}\right) \leq \frac{h^*}{2}\}$ and let N be the smallest positive integer such that

$$\frac{\gamma^*}{2} + Na \geq \frac{h^*}{2} \quad (3.7)$$

Set $t_k = \sigma + kT^*$, $k = 0,1,2, \ldots \ldots .., N$ where T^* is to be determined later and will be independent of σ and $\emptyset$

We claim that

$$V(t) \leq \frac{\gamma^*}{2} + (N - k)a, \text{ for } t \geq t_k,\ k = 0,1,2, \ldots, N \quad (3.8)$$

Obviously, (3.8) holds for $k = 0$ in view of (3.5) and (3.7). Suppose that for some $k: 0 \leq k \leq N - 1$, (3.8) holds. We want to show that (3.8) also holds for $k + 1$, i.e.

$$V(t) \leq \frac{\gamma^*}{2} + (N - k - 1)a, \text{ for } t \geq t_{k+1} \quad (3.9)$$

Let $r = \max\left\{l, q\left(\frac{\gamma^*}{2}\right)\right\}$. We first prove that if there exist some $t_1 \geq t_k + r$ with

$$V(t_1) \leq \frac{\gamma^*}{2} + (N - k - 1)a, \quad (3.10)$$

Then

$$V(t) \leq \frac{\gamma^*}{2} + (N - k - 1)a, \text{ for } t \geq t_1 \quad (3.11)$$

In fact, suppose it is not true, then there is a $\hat{t} > t_1$ with $V(\hat{t}) > \frac{\gamma^*}{2} + (N-k-1)a$ and $V'(\hat{t}) > 0$

Since $\frac{\gamma^*}{2} \le V(\hat{t}) \le \frac{h^*}{2}$ and $\hat{t} > t_1 \ge t_k + r,$

we have $P(V(\hat{t})) > V(\hat{t}) + a > \frac{\gamma^*}{2} + (N-k)a \ge V\big(t - \tau(t)\big)$

Noting that $V(\hat{t}) > \frac{\gamma^*}{2}$ and $q(s)$ is non increasing,

we have $q(\,V(\hat{t})) \le q(\frac{\gamma^*}{2}),$ and thus

$$\hat{t} - q\big(V(\hat{t})\big) \ge \hat{t} - q(\frac{\gamma^*}{2}) \ge \hat{t} - r.$$

Hence there holds

$$P\big(V(\hat{t})\big) > V\big(t - \tau(t)\big)$$

It follows from (3.3) or (3.4) that either

$$V'_1(\hat{t}) \le -W_{51}(\left|x^{(1)}(\hat{t})\right| \le 0$$

Or $$V'_2(\hat{t}) \le -W_{52}(\left|x^{(2)}(\hat{t})\right| \le 0$$

In either case it leads to a contradiction. This shows that (3.11) holds.

Next, we show that there does exist some $t_1 \in [t_k + r, t_{k+1}]$ such that (3.10) holds.

Suppose not, for all $t \ge t_k + r$ we would have

$$\frac{\gamma^*}{2} + (N-k-1)a < V(t) \le \frac{\gamma^*}{2} + (N-k)a \tag{3.12}$$

This with the same arguments as above we obtain for $t \ge t_k + r$

Either $$V'(t) = V'_1(t) \le -W_{51}(\left|x^{(1)}(t)\right| \tag{3.13}$$

or $$V'(t) = V'_2(t) \le -W_{52}(\left|x^{(2)}(t)\right| \tag{3.14}$$

As we have shown before, one or the other inequalities holds on successive subintervals of $[t_k + r, +\infty)$. For any $t > t_k + r$ we denote on $[t_k + r, t]$ by I1, the set of subintervals where $V_1(t) \ge V_2(t)$ and by I2 the set of subintervals where $V_1(t) \le V_2(t)$. Then on I1, where (3.13) holds we have by (3.6) with j=1 and (3.12) that

$$W_{21}\big(\left|x^{(1)}(t)\right|\big) + W_{31}\left[{W_{31}}^{-1}\frac{\left(\frac{\gamma^*}{4}\right)}{2} + K_1\int_{t-l}^{t} W_{41}\left(\left|x^{(1)}(s)\right|\right)ds\right] \ge V_1(t) = V(t) > \frac{\gamma^*}{2}$$

Which implies that either

$$W_{21}\left(\left|x^{(1)}(t)\right|\right) \geq \frac{\gamma^*}{4} \qquad \text{i.e.} \qquad \left|x^{(1)}(t)\right| \geq {W_{21}}^{-1}\left(\frac{\gamma^*}{4}\right) \tag{3.15}$$

Or $$W_{31}\left[{W_{31}}^{-1}\frac{\left(\frac{\gamma^*}{4}\right)}{2} + K_1 \int_{t-l}^{t} W_{41}\left(\left|x^{(1)}(s)\right|\right)ds\right] \geq \frac{\gamma^*}{4}$$

i.e. $$\int_{t-l}^{t} W_{51}\left(\left|x^{(1)}(s)\right|\right)ds \geq {W_{31}}^{-1}\frac{\left(\frac{\gamma^*}{4}\right)}{(2K_1)}$$

Then by lemma 1 there exist a $\beta_1 > 0$ such that

$$\int_{t-l}^{t} W_{51}\left(\left|x^{(1)}(s)\right|\right)ds \geq \beta_1 \tag{3.16}$$

Let $E_{11} = \{t \in I_1 | W_{21}\left(\left|x^{(1)}(t)\right|\right) \geq \frac{\gamma^*}{4}\}$ and $E_{21} = [t_k + r, t] - E_{11}$.

Similarly on $I_2$where (3.14) holds, we have by (3.6) with j=2 and (3.12) that either

$$\left|x^{(2)}(t)\right| \geq {W_{22}}^{-1}\left(\frac{\gamma^*}{4}\right) \tag{3.17}$$

Or for some $\beta_2 > 0$,

$$\int_{t-l}^{t} W_{52}\left(\left|x^{(2)}(s)\right|\right)ds \geq \beta_2 \tag{3.18}$$

Let $E_{12} = \left\{t \in I_2 \middle| W_{22}\left(\left|x^{(2)}(t)\right|\right) \geq \frac{\gamma^*}{4}\right\}$ and $E_{22} = [t_k + r, t] - E_{12}$

Suppose ${k_j}^*$ is the positive integer with

$${k_j}^*\beta_j > \frac{h^*}{2} \geq \left({k_j}^* - 1\right)\beta_j, \quad j = 1,2$$

Let $T^* = r + {k_1}^* l + {k_2}^* l + \frac{h^*}{W_{51}\left[{W_{21}}^{-1}\left(\frac{\gamma^*}{4}\right)\right]} + \frac{h^*}{W_{52}\left[{W_{22}}^{-1}\left(\frac{\gamma^*}{4}\right)\right]}$ and set $t = t_k + T^*$

Since the total measure of the interval $[t_k + r, t_k + T^*]$ is

$${k_1}^* l + {k_2}^* l + \frac{h^*}{W_{51}\left[{W_{21}}^{-1}\left(\frac{\gamma^*}{4}\right)\right]} + \frac{h^*}{W_{52}\left[{W_{22}}^{-1}\left(\frac{\gamma^*}{4}\right)\right]},$$

there must hold at least one of the following cases:

a) $m(E_{11}) \geq \frac{h^*}{W_{51}\left[{W_{21}}^{-1}\left(\frac{\gamma^*}{4}\right)\right]}$;

b) $m(E_{21}) \geq {k_1}^* l$

c) $m(E_{12}) \geq \frac{h^*}{W_{52}\left[W_{22}{}^{-1}\left(\frac{\gamma^*}{4}\right)\right]}$;

d) $m(E_{22}) \geq k_2{}^* l$, where $m(E_{ij})$ denote the measure of set $E_{ij}(i,j = 1,2)$

If (a) holds, then it follows from (3.13) and (3.15) that

$$V_1(t_k + T^*) \leq V_1(t_k + w) - \int_{t_k+w}^{t_k+T^*} W_{51}\left(\left|x^{(1)}(s)\right|\right)ds$$

$$\leq \frac{h^*}{2} - \int_{E_{11}} W_{51}\left[W_{21}{}^{-1}\left(\frac{\gamma^*}{4}\right)\right]ds \leq -\frac{h^*}{2} < 0.$$

This is a contradiction.

If (b) holds then there must exist $k_1{}^*$ points in E_{21} such that

$\widehat{t_1} < \widehat{t_2} < \cdots \ldots . < \widehat{t_{k_1{}^*}}$, $\widehat{t_1} \geq t_k + w + l$ and $\widehat{t_\iota} \geq \widehat{t_{\iota-1}} + l$, $i = 1,2, \ldots\ldots, k_1{}^*$

Hence, we have by (3.13) and (3.16) that

$$V_1(t_k + T^*) \leq V_1(t_k + w) - \int_{t_k+w}^{t_k+T^*} W_{51}\left(\left|x^{(1)}(s)\right|\right)ds$$

$$\leq \frac{h^*}{2} - \sum_{i=1}^{k_1{}^*} \int_{\widehat{t_\iota}-l}^{\widehat{t_\iota}} W_{51}\left(\left|x^{(1)}(s)\right|\right)ds \leq \frac{h^*}{2} - k_1{}^*\beta_1 < 0$$

Again, a contradiction

Similarly, by (3.14) and (3.17) or (3.18), we can conclude that (c) or (d) also leads to a contradiction. This shows that there must exist some $\hat{t} \in [t_k + w, t_k + T^*]$ such that (3.10) holds. Thus

$$V(t) \leq \frac{\gamma^*}{2} + (N - k - 1)a, \text{ for all } t \geq t_k + T^* = t_{k+1},$$

i.e, (3.9) is true.

By induction, we arrive at

$$\frac{\left[W_{11}\left(\left|x^{(1)}(t)\right|\right) + W_{12}\left(\left|x^{(2)}(t)\right|\right)\right]}{2} \leq V(t) \leq \frac{\gamma^*}{2} \quad \text{for } t \geq t_N = \sigma + NT^* \qquad (3.19)$$

If $V_1(\tau_l) \geq V_2(\tau_l)$ then $V(\tau_l) = V_1(\tau_l)$; from inequality (3.19) and condition (iii) we have

$$V(\tau_l) = V\left(\tau_l, x(\tau_l{}^-) + I_k\left(x(\tau_l{}^-)\right)\right) = V_1\left(\tau_l, x(\tau_l{}^-) + I_k\left(x(\tau_l{}^-)\right)\right)$$

$$\leq (1 + b_l)V_1\left(\tau_l{}^-, x(\tau_l{}^-)\right) \leq (1 + b_l)\frac{h^*}{2}$$

If $V_1(\tau_l) < V_2(\tau_l)$ then $V(\tau_l) = V_2(\tau_l)$; from inequality (3.19) and condition (iii) we have

$$V(\tau_l) = V\left(\tau_l, x(\tau_l^{-}) + I_k\big(x(\tau_l^{-})\big)\right) = V_2\left(\tau_l, x(\tau_l^{-}) + I_k\big(x(\tau_l^{-})\big)\right)$$

$$\leq (1 + b_l)V_2\big(\tau_l^{-}, x(\tau_l^{-})\big) \leq (1 + b_l)\frac{h^*}{2}$$

So in either case, we have proved that $V(\tau_l) \leq (1 + b_l)\frac{h^*}{2}$

Next we prove that

$$V(t) \leq (1 + b_l)\frac{h^*}{2} \text{ for } \tau_l \leq t < \tau_{l+1} \tag{3.20}$$

If inequality (3.20) does not hold, then there is a $\hat{s} \in (\tau_l, \tau_{l+1})$ such that

$$V(\hat{s}) > (1 + b_l)\frac{h^*}{2}$$

and $$V'(\hat{s}) > -W_{51}\big(\big|x^{(1)}(\hat{s})\big|\big),\ \ V(t) \leq V(\hat{s}) \text{ for } t \in [\tau_l, \hat{s}\,]$$

Since $t \geq \tau(t) \geq 0$, we have

$$V(\hat{s} - \tau(\hat{s})) \leq P(V(\hat{s}))$$

From (3.4) $\quad V'(\hat{s}) \leq -W_{51}\big(\big|x^{(1)}(\hat{s})\big|\big)$.

This is a contradiction. So (3.20) holds.

If $\quad V_1(\tau_{l+1}) \geq V_2(\tau_{l+1})$ then $V(\tau_{l+1}) = V_1(\tau_{l+1})$;

from inequality (3.20) and condition (iii) we have

$$V(\tau_{l+1}) = V\left(\tau_{l+1}, x(\tau_{l+1}^{-}) + I_k\big(x(\tau_{l+1}^{-})\big)\right)$$

$$= V_1\left(\tau_{l+1}, x(\tau_{l+1}^{-}) + I_k\big(x(\tau_{l+1}^{-})\big)\right)$$

$$\leq (1 + b_{l+1})V_1\big(\tau_{l+1}^{-}, x(\tau_{l+1}^{-})\big) \leq (1 + b_{l+1})(1 + b_l)\frac{h^*}{2}$$

If $\quad V_1(\tau_{l+1}) < V_2(\tau_{l+1})$ then $V(\tau_{l+1}) = V_2(\tau_{l+1})$;

from inequality (3.20) and condition (iii) we have

$$V(\tau_{l+1}) = V\left(\tau_{l+1}, x(\tau_{l+1}^{-}) + I_k\big(x(\tau_{l+1}^{-})\big)\right)$$

$$= V_2\left(\tau_{l+1}, x(\tau_{l+1}^{-}) + I_k\big(x(\tau_{l+1}^{-})\big)\right)$$

$$\leq (1 + b_{l+1})V_2\big(\tau_l^{-}, x(\tau_l^{-})\big) \leq (1 + b_{l+1})(1 + b_l)\frac{h^*}{2}$$

So in either case, we have proved that

$$V(\tau_{l+1}) \leq (1+b_{l+1})(1+b_l)\frac{h^*}{2}$$

By simple induction, we can prove that in general

$$V(t) \leq (1+b_{l+i+1}) \ldots\ldots (1+b_l)\frac{h^*}{2} \quad \text{for } \tau_{l+i} \leq t \leq \tau_{l+i+1}$$

Taking this together with (3.2) and (3.19) and $\prod_{k=1}^{\infty}(1+b_k) = M$, we have

$$\frac{[W_{11}(|x^{(1)}(t)|)+W_{12}(|x^{(2)}(t)|)]}{2} \leq V(t) \leq M\frac{h^*}{2} \text{ for } t \geq \sigma$$

Since $Mh^* = min\{w_{11}(h), w_{12}(h)\}$, we have

$$W_{11}(|x^{(1)}(t)|) \leq W_{11}(h), W_{12}(|x^{(2)}(t)|) \leq W_{12}(h).$$

Therefore,

$$|x(t)| = max(|x^{(1)}(t)|), (|x^{(2)}(t)|) \leq \gamma, \qquad \text{for } t \geq \sigma + T$$

where $T = NT^*$ is obviously independent of σ and $\emptyset$.

Therefore, the zero solution of (2.1) is U.A.S

Theorem 3.2: Suppose that there exist continuous lyapunov functionals $V_j: [\alpha, \infty) \times {C_H}^j(t) \to R^+ (j = 1,2, \ldots, m)$ satisfying (i) in Theorem 3.1 and such that

(ii)' when $V_k\left(t, x^{(k)}(.)\right) = \max[V_j(t, x^j(.))|1 \leq j \leq m)]$, there holds ${V_k}'(t, x^{(k)}(.)) \leq -W_{5k}(|x^{(k)}(t)|$ if $V_k(s, x^{(k)}(.)) \leq P_k(V_k(t, x^{(k)}(.))$ for $s \in [\max\{\alpha, t - q_k(v_k(t))\}, t]$, where W_{5j} are wedge functions and P_j, q_j have the same properties as in Theorem 3.1 for $j = 1,2, \ldots, m$ then the zero solution of (2.1) is U.A.S

4. CONCLUSION

In this work, we have considered the impulsive infinite delay differential equations. By using Lyapunov functions and Razumikhin technique, we have obtained some more general results. When using the razumikhin technique, we used a new technique given in [4], this technique has been extended to study impulsive infinite delay differential systems.

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