Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 9, Number 2 (2017), pp. 133-138

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Hamilton's Canonical Equations for a Classical System with Velocity Dependent Potential Energy-a Mathematical Study

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Abstract

In reality, most of the classical systems are holonomic conservative system for which potential energy is only function of generalised coordinates but not generalised velocity. For system with this kind of potential energy, Hamilton's canonical equations are very usual and with these equations one can show that for N particle holonomic system, if the instantaneous position of any constituent particle is not explicit function of time then Hamiltonian of that system i.e. the total energy of that system will be conserved. In this mathematical study I have tried to construct the forms of Hamilton's canonical equations for the classical system with velocity dependent potential energy. Although such kind of system with generalised potential energy is not very abundant in reality, but a successful mathematical study is made by me in this work with two cases – charge particle motion in electromagnetic field and particle motion in non inertial rotating frame and then Hamilton's canonical equations for velocity dependent potential energy are constructed which will be very helpful for analysis for similar classical systems.

Keywords: Holonomic system, Canonical equations, Velocity dependent potential energy, Electromagnetic field, Non inertial rotating frame.

1. INTRODUCTION

In classical mechanics, we have for a holonomic conservative system; the potential energy is in general a function of generalized coordinate's i.e. $\boldsymbol{V = V(q_j)}$ and it may or may not depend of time explicitly but $\boldsymbol{V \neq V(\dot{q}_j)}$ whereas, as far as kinetic energy is concerned, $\boldsymbol{T = T(q_j, \dot{q}_j)}$. Thus for that system we have Lagrangian [1] of the system $\boldsymbol{L(q_j, \dot{q}_j) = T(q_j, \dot{q}_j) - V(q_j)}$

And the Hamiltonian of that system is $\boldsymbol{H(q_j, p_j) = \sum p_j \dot{q}_j - L(q_j, \dot{q}_j)}$

This Hamiltonian will be equal to the total energy of that system if the ith particle position of that N particle system is not explicit function of time and kinetic energy is homogeneous function of generalized velocity in degree 2.

With respect to this Hamiltonian of holonomic conservative system we have well known Hamilton's canonical equations which are $\boldsymbol{\frac{\partial H}{\partial p_j} = \dot{q}_j}$ and $\boldsymbol{\frac{\partial H}{\partial q_j} = -\dot{p}_j}$

But we are now interested about these canonical equations for a velocity dependent potential [2] which is generalized potential $\boldsymbol{U(q_j, \dot{q}_j)}$.

2. MOTION OF CHARGE PARTICLE IN ELECTRO MAGNETIC FIELD

Let us now consider the case of charge particle motion [3] in electromagnetic field. Here Lagrangian for such particle motion will be

$$\boldsymbol{L = T - U = \frac{1}{2} m v^2 - q(\varphi - \vec{v}.\vec{A}) = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - q\left(\varphi - \sum \dot{x}_j A_j\right)}$$

Now for kinetic energy term $\boldsymbol{T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}$,

We have $\boldsymbol{\frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x}$, $\boldsymbol{\frac{\partial T}{\partial \dot{y}} = m\dot{y} = p_y}$, $\boldsymbol{\frac{\partial T}{\partial \dot{z}} = m\dot{z} = p_z}$**.** Thus we get the Hamiltonian of that charge particle motion as

$$\boldsymbol{H = \sum p_j \dot{q}_j - L = (\dot{x}p_x + \dot{y}p_y + \dot{z}p_z) - L = \frac{1}{m}(p_x^2 + p_y^2 + p_z^2) - L = \frac{p^2}{m} - \left[\frac{p^2}{2m} - q(\varphi - \vec{v}.\vec{A})\right]}$$

Hence finally, $$\boldsymbol{H = \frac{p^2}{2m} + q(\varphi - \vec{v}.\vec{A}) = \frac{p^2}{2m} + q\left(\varphi - \frac{(\vec{p}.\vec{A})}{m}\right) = T + U}$$

With respect to this Hamiltonian of charge particle motion we get

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{\mathrm{m}} - \frac{\mathrm{q}}{\mathrm{m}} A_x = \dot{x} - \frac{\mathrm{q}}{\mathrm{m}} \frac{\partial}{\partial p_x}(p_x A_x) = \dot{x} - \mathbf{q}\frac{\partial}{\partial p_x}\left(\frac{p_x A_x}{m}\right) = \dot{x} - \mathbf{q}\frac{\partial}{\partial p_x}\left(\frac{(\vec{p}.\vec{A})}{m}\right)$$

This can also be rewritten as

$$\frac{\partial H}{\partial p_x} = \dot{x} + q\frac{\partial}{\partial p_x}\left(\varphi - \frac{(\vec{p}.\vec{A})}{m}\right) = \dot{x} + \frac{\partial}{\partial p_x}\left[q\left(\varphi - \frac{(\vec{p}.\vec{A})}{m}\right)\right] = \dot{x} + \frac{\partial \mathrm{U}}{\partial p_x} \quad \left[\text{Since } \frac{\partial \varphi}{\partial p_x} = 0\right]$$

Thus in general we have for velocity dependent potential $\frac{\partial H}{\partial p_j} = \dot{q}_j + \frac{\partial \mathrm{U}}{\partial p_j}$

On the other hand, $\frac{\partial H}{\partial x} = q\frac{\partial}{\partial x}\left(\varphi - \frac{(\vec{p}.\vec{A})}{m}\right) = \frac{\partial}{\partial x}\left[q\left(\varphi - \frac{(\vec{p}.\vec{A})}{m}\right)\right] = \frac{\partial U}{\partial x}$

Again in general we have $\frac{\partial H}{\partial q_j} = \frac{\partial \mathrm{U}}{\partial q_j}$

So for the motion of a charged particle in external electromagnetic field, Hamilton's canonical equation for velocity dependent potential should be written as

$$\frac{\partial H}{\partial p_j} = \dot{q}_j + \frac{\partial \mathrm{U}}{\partial p_j} \quad \text{and} \quad \frac{\partial H}{\partial q_j} = \frac{\partial \mathrm{U}}{\partial q_j} = -\left[Q_j - \frac{d}{dt}\left(\frac{\partial \mathrm{U}}{\partial \dot{q}_j}\right)\right] \text{----------------------------------(1)}$$

Usually if we now turn back to the holonomic conservative system $U \neq U(p_j)$ *i.e.* $U \neq U(\dot{q}_j)$ we have $U = V(q_j)$ and then we get for $T = T(\dot{q}_j)$ and $V = V(q_j)$

$\frac{\partial H}{\partial p_j} = \dot{q}_j$ and $\frac{\partial H}{\partial q_j} = -Q_j = +\frac{\partial \mathrm{V}}{\partial q_j} = -\frac{\partial(\mathrm{T}-\mathrm{V})}{\partial q_j} = -\frac{\partial \mathrm{L}}{\partial q_j} = -\frac{d}{dt}\left(\frac{\partial \mathrm{L}}{\partial \dot{q}_j}\right) = -\dot{p}_j$ which are our well known Hamilton's canonical equation for holonomic conservative system.

3. MOVING PARTICLE IN NON INERTIAL ROTATING FRAME

Let us now consider the motion of a particle or system in a non inertial rotating frame [4]. For particle motion in a rotating frame, the total or effective force acting on that particle is given by

$\vec{F}]_{total} = [\vec{F}]_{actual} + [\vec{F}]_{centrifugal} + [\vec{F}]_{coriolis} + [\vec{F}]_{Eular}$. Mathematically it is given by

$$\vec{F}]_{total} = [\vec{F}]_{actual} + \{[-m\vec{\omega} \times (\vec{\omega} \times \vec{r})] + [-2m(\vec{\omega} \times \vec{v})] + \left[-m\left(\frac{d\vec{\omega}}{dt} \times \vec{r}\right)\right]$$

where $\vec{v}$ is the velocity of that moving particle in that rotating frame.

By applying the rules of vector analysis, this force equation is also given by

$$\vec{F}]_{total} = [\vec{F}]_{actual} + \left[\frac{m}{2}\vec{\nabla}|\vec{\omega}\times\vec{r}|^2 + \left\{-m\frac{\partial}{\partial t}(\vec{\omega}\times\vec{r})\right\} + m\dot{\vec{r}}\times\left\{\vec{\nabla}\times(\vec{\omega}\times\vec{r})\right\}\right].$$

This equation can also be expressed as

$$\vec{F}]_{total} = [\vec{F}]_{actual} + m\left[-\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t} + \vec{v}\times\left(\vec{\nabla}\times\vec{A}\right)\right] \text{-------------------------------}(2)$$

where the corresponding scalar and vector potentials [5-7] are given by $\varphi = -\frac{1}{2}|\vec{\omega}\times\vec{r}|^2$ and $\vec{A} = \vec{\omega}\times\vec{r}$

Surprisingly we see that the 2nd part of this force equation is very similar to the effective force on a moving charge in electromagnetic field. Thus we can take all these cases of charge particle motion in electromagnetic field, particle motion in non inertial rotating frame, ... at the same platform of velocity dependent potential for which Hamilton's canonical equations, as given by the equation (1) will equally be hold.

4. PARTICLE WITH VELOCITY DEPENDENT POTENTIAL ENERGY IN ROTATING FRAME

In reference of the force equation (2), the corresponding velocity dependent potential for particle motion in rotating frame will be

$$U = V + m\left[-\frac{1}{2}|\vec{\omega}\times\vec{r}|^2 - \dot{\vec{r}}.(\vec{\omega}\times\vec{r})\right] \quad \text{when} \quad \vec{F}]_{total} = -\vec{\nabla}U \ \text{and}\ [\vec{F}]_{actual} = -\vec{\nabla}V$$

For this velocity dependent potential U we have the Hamiltonian of the system

$$H = \frac{p^2}{2m} + U = \frac{p^2}{2m} + \left[V + \left\{-\frac{m}{2}|\vec{\omega}\times\vec{r}|^2\right\} + \left\{-\vec{p}.(\vec{\omega}\times\vec{r})\right\}\right].$$

With respect to this Hamiltonian we get

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} - (\vec{\omega}\times\vec{r})_x = \dot{x} + \frac{\partial}{\partial p_x}\left[V + \left\{-\frac{m}{2}|\vec{\omega}\times\vec{r}|^2 - \vec{p}.(\vec{\omega}\times\vec{r})\right\}\right] = \dot{x} + \frac{\partial U_o}{\partial p_x}$$

since $\frac{\partial V}{\partial p_x} = 0$

Thus in general we have $\frac{\partial H}{\partial p_j} = \dot{q}_j + \frac{\partial U}{\partial p_j}$ which is already obtained for charged particle motion in electromagnetic field as given in equation (1)

On the other hand, also for this case of particle motion in rotating frame

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left\{-\frac{m}{2}|\vec{\omega}\times\vec{r}|^2\right\} + \frac{\partial}{\partial x}\{-\vec{p}.(\vec{\omega}\times\vec{r})\}$$

But mathematically we have $\frac{\partial}{\partial x}\left\{-\frac{m}{2}|\vec{\omega}\times\vec{r}|^2\right\} = m[\vec{\omega}\times(\vec{\omega}\times\vec{r})]_x$ and

$$\frac{\partial}{\partial x}\{-\vec{p}.(\vec{\omega}\times\vec{r})\}$$

$$= 2m(\vec{\omega}\times\vec{v})_x - m\frac{d}{dt}(\vec{\omega}\times\vec{r})_x + m(\frac{d\vec{\omega}}{dt}\times\vec{r})_x = 2m(\vec{\omega}\times\vec{v})_x - m\frac{d}{dt}(v)_x + m(\frac{d\vec{\omega}}{dt}\times\vec{r})_x$$

$$= 2m(\vec{\omega}\times\vec{v})_x - F_x + m(\frac{d\vec{\omega}}{dt}\times\vec{r})_x$$

So finally we get

$$\frac{\partial H}{\partial x} = m[\vec{\omega}\times(\vec{\omega}\times\vec{r})]_x + 2m(\vec{\omega}\times\vec{v})_x - F_x + m(\frac{d\vec{\omega}}{dt}\times\vec{r})_x$$

$$= -\{F_x + [-m[\vec{\omega}\times(\vec{\omega}\times\vec{r})]_x] + [-2m(\vec{\omega}\times\vec{v})_x] + [-m(\frac{d\vec{\omega}}{dt}\times\vec{r})_x]\}$$

$$= -\{F_x + [\vec{F}_{centrifugal}]_x + [\vec{F}_{corilis}]_x + [\vec{F}_{Eular}]_x] = \frac{\partial U}{\partial x} = -F_x]_{total} = -\dot{p}_x$$

So in general we have $\frac{\partial H}{\partial q_j} = \frac{\partial U}{\partial q_j}$ which is also obtained for charged particle motion in electromagnetic field as given in equation (1)

5. CONCLUSION

From this whole theoretical study we can conclude that although in most of the cases, the classical systems are holonomic systems with velocity independent potential energy of the system and for those systems Hamilton's canonical equations are given by $\frac{\partial H}{\partial p_j} = \dot{q}_j$ and $\frac{\partial H}{\partial q_j} = -\dot{p}_j$, but for classical system with velocity dependent potential energy, Hamilton's canonical equations are given by $\frac{\partial H}{\partial p_j} = \dot{q}_j + \frac{\partial U}{\partial p_j}$ and $\frac{\partial H}{\partial q_j} = \frac{\partial U}{\partial q_j}$. These are theoretically supported by charge particle motion in electromagnetic field, by the motion of particle in non inertial frame … etc and hopefully I think that these equations will help for analysis the similar classical system with velocity dependent potential energy or generalised potential energy.

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