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The Bi-Level Control Policy for $M^X{}_{i(m,N)}$/M/1/BD/SV Queueing System Under Restrictive Admissibility

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Abstract

Queueing systems with Bi-level control policy have been studied in the literature (Lee et al. 2003,1994 and J.C.Ke 2004). Most of the queueing models Bi Level control policy can be analyzed using Decomposition property. The objective of this paper are i) to analyze some general bulk arrival queueing system with server vacations and early setup time and to drive the probability generating functions ii) Study state various performance measures and find the optimal values

Keywords: Setup time, Bi-level, Bulk arrival, working vacations, Restrictive Admissibility

I INTRODUCTION

The bi-level control policy was first introduced by Lee and Park (1997) for an M/G/1queueing system. They have used the decomposition property of vacation queues to derive the distribution of the number of units in the system and developed a procedure to find the optimal value of (m,N) that minimizes a linear cost. They have shown that the double-threshold policy is more beneficial than the conventional single threshold policy. Lee *et al*., (1998) analyzed M^X/M/1 queue with bi-level control and obtained the queue length and waiting time distribution.

Later Lee *et al*.,(2003) extended Lee and Park (1997) model to a non-Markovian batch arrival system with/without server's vacation .The works mentioned above focused only on reliable servers and do not investigate the cases involving unreliable server with both vacations and early startup. Ke (2004a) considered a bi-level control of batch arrival $M^X/G/1$ queueing system in which the system is unreliable and is characterized by an early setup and multiple vacations. All these papers used the well known decomposition property of vacation queues directly to derive the PGF of the stationary queue length distribution.

Past work regarding queues may be divided into two categories (i) the case of controlling the service and (ii) the case of controlling the arrivals. Regarding the control policy of service, Yadin and Naor (1963) introduced an N-policy for M/M/1 queueing system, which turns the server on whenever N (predetermined value) or more customers present in the system and turns off the server when the system becomes empty. Lee *et al*., (1994a) successively combined the batch arrival queues with N-policy and later Lee and Srinivasan (1989), Lee *et al*., (1994b and 1995) studied the behavioral characteristics of batch arrival queues with N-policy and server vacations. But these research works do not involve setup operations.

In many real world production systems, setup operations are needed in several occasions. For example, when the machine changes its production type, the operator of the machine changes the tools and adjusts the machine speed. Sometimes, a setup operation takes several days and is very costly. One way to reduce the setup cost per unit time is to delay the production until some number of raw materials accumulates. But when the setup cost is very high, the operator may not need to wait until the accumulated items reach the usual single threshold N (i.e.) the sever can start the setup operation when m ($m \leq N$) customers accumulate in the queue. And after the setup, if there are less than N customers in the queue, then the server remains dormant in the system until the number of customers reaches N. If N (or) more customers are in the system, after the setup, the server begins to serve the customers immediately. This policy is called bi-level threshold policy (or) (m, N) policy. This policy is more general than the usual (N, N) policy in which the server starts a setup when N customers have piled up in the queue and then starts his service as soon as the setup is complete.

In the literature of queueing, most of the papers deal with the queueing system wherein the service stations are reliable (i.e.) do not fail. However in practice, we often meet the case where the service station may fail and can be repaired. The performance of the system may be heavily affected by the server break downs and limited repair capacity. On such situations queueing system with unreliable service stations are worth to investigate from performance prediction view point.

Regarding the control policy of arrivals many authors (Rue and Roshen Shine, 1981; Netus, 1984 and Stidham, 1985) deal with the policy, where not all arriving batches are allowed to join the system at all times. Such restrictions may be necessary in many real life situations, particularly in the over saturated queue with arrivals occurring faster than the departures.

Ke and Pearn (2004) discussed the optimal management policy for heterogeneous arrival queueing systems with server breakdown and multiple vacations for an M/M/1 queue, and derived the system size distribution and employed the PGF to obtain the system characteristics. But the concept of heterogeneous arrivals is not combined with the double threshold policy, batch arrival queues under server breakdowns and vacations in literature.

In this Chapter, batch arrival Markovian queueing systems along with server breakdowns, bi-level threshold policy for service and restricted admissibility policy for arrivals are analyzed under single. The PGF of the system size is obtained through the Chapman-Kolmogorov balanced equations satisfied by the steady state system size probabilities. The PGF is presented in closed form so that various performance measures can be calculated easily. A cost model and a procedure to find the optimal values of the decision variables m and N that minimize the linear average cost are developed.

II MODEL DESCRIPTION

Customers arrive in batches in accordance with the time homogeneous Poisson process with group arrival rate λ. The batch size X is a random variable with probability distribution $Pr(X = k) = g_k, k=1,2,3,\ldots$(i.e) the probability that a batch of k units, arrive in an infinitesimal interval (t, t+h) is $\lambda g_k+o(h)$.Not all arriving batches are allowed to join the system at all times. The probability that an arriving batch is allowed to join the system varies according to the system state which falls into one of the 3-categories namely idle or busy or break down period. r_1 $(0 \le r_1 \le 1)$ denotes the probability that an arriving batch is allowed to join the system while the server is idle and r_i $(i = 2, 3)$ $(0 \le r_i \le 1)$ respectively denotes the probability with which an arriving batch joins the system during the busy and break down periods of the server. The customers who arrive and join the system form a single waiting line based on the order of the batches. It is further assumed that the customers with in a batch are pre-ordered for service. The customer is served one by one according to the order in the queue. A cycle starts whenever the system empties; the server is deactivated and leaves the system for a vacation of random length V, following an exponential distribution of parameter η.After returning from the vacation, if the server finds m (or) more customers in the system, then the server immediately starts a setup operation of random length D. Otherwise the server joins the system and remains idle in the system until the system size reaches atleast m and then starts the setup work (i.e.) single vacation policy is adopted. The period during which the server remains idle in the system before starting the setup work is called buildup period. The setup time is assumed to be an exponentially distributed random variable with mean $(1/\gamma)$.

At the end of the setup period, if the queue length is greater than or equal to N, then the server begins to serve the customers, one at a time. Otherwise the server remains idle (dormant) in the system waiting for the queue length to reach atleast N, to start the service. The service time of each customer is an independent and identically distributed random variable, following exponential distribution $(1 - e^{-\mu t})$.

The server is subject to break downs at any time while working, with Poisson rate α. Whenever the system fails, the server is sent immediately for repair at a repair facility, where the repair time is independent and identically distributed random variable B_r following an exponential distribution $(1 - e^{-\beta t})$. The customer, who is just being served when the server breakdown, joins the head of the waiting line and resumes the service as soon as the server returns from the repair facility. This type of service continues until the system becomes empty again. Thus vacation period, buildup period, setup period and dormant period together represent an idle period and the sum of busy period and the break down period gives the completion period. Thus an idle period and completion period contribute a cycle. This model is denoted by $M^X{}_{i(m,N)}/M/1/BD/SV$ in which BD denotes breakdown and SV denotes single vacation. The control policy adopted for service is called (m,N) policy (double threshold policy or bi-level policy). In this model, the first threshold m, is used to control the starting condition of a setup operation, and the second threshold N is used to control the starting condition of service. If m = N, the model becomes the usual setup time queueing model with N-policy and vacations. It is also assumed that all the stochastic processes involved in the model are independent of each other.

To derive the PGF of the system size distribution the following notations are used to write the steady state system size equations.

m, N	:	Bi-level thresholds
λ	:	group arrival rate
X	:	Group size random variable
Pr (x = k)	:	g_k $(k \geq 1)$
X(z)	:	$\sum_{k=1}^{\infty} g_k z^k$ the PGF of X

$g_n^{(i)}$: i-fold convolution of g_n's with itself, where

$$g_n^{(0)} = 1, \quad \text{if } n = 0$$
$$= 0, \quad \text{if } n > 0.$$

r_i (i=1 to 3) : The probability that the arriving batch is allowed to join the system while the server is in idle (buildup, setup, dormant and vacation) busy and in breakdown state respectively.

$\lambda_i = \lambda r_i$ i = 1,2,3 : The probability that the arriving batch of size i, joins the system while the server is idle (buildup, setup, dormant and vacation) busy and in break down state respectively.

N(t) : The number of customers in the system at time t, including the one in service.

Let μ, η, γ and β respectively denote the parameters of the exponential distributions of the random variables namely service time (S), vacation time (V), setup time (D) and repair time (B_r). The Laplace Stieltes transforms of the distributions of the random variables respectively are given by :

$$S^*(\theta) = \mu/\mu+\theta; \quad V^*(\theta) = \eta/(\eta+\theta); D^*(\theta) = \lambda/(\lambda+\theta) \text{ and } B_r^*(\theta) = \beta/(\beta+\theta).$$

The server's states at time t are denoted by the random variableΩ.

Ω { = 0 if the server is vacation;1 if the server is bulid up; 2 if the server is setup;3 if the server is dormant; 4 if the server is busy ; 5 if the server is breakdown }

The probability that there are n customers in the system and the server is in vacation, buildup, setup, dormant, busy and in break down states respectively at time t are defined by:

$Q_n(t) = \Pr(N(t) = n, \Omega = 0)\ n \geq 0$

$R_n(t) = \Pr(N(t) = n, \Omega = 1)\ 0 \leq n \leq m-1$

$D_n(t) = \Pr(N(t) = n, \Omega = 2), \quad n \geq m$

$U_n(t) = \Pr(N(t) = n, \Omega = 3), \quad m \leq n \leq N-1$

$P_n(t) = \Pr(N(t) = n, \Omega = 4), \quad n \geq 1$

$B_n(t) = \Pr(N(t) = n, \Omega = 5), \quad n \geq 1.$

Then (N(t), Ω) follows a Markov process.

Further let Q_n, R_n, D_n, U_n, P_n and B_n denote the respective steady state probabilities (independent of time t) are given by:

(i.e) $Q_n = \lim_{t\to\infty} Q_n(t)$; $R_n = \lim_{t\to\infty} R_n(t)$; $D_n = \lim_{t\to\infty} D_n(t)$; $U_n = \lim_{t\to\infty} U_n(t)$; $P_n = \lim_{t\to\infty} P_n(t)$;
$B_n = \lim_{t\to\infty} B_n(t)$.

III THE SYSTEM SIZE DISTRIBUTIONS:

Observing the changes of the states during the interval $(t, t+\Delta t)$ at any time t, the forward set of Kolmogrov equations satisfied by the steady state probabilities are given by:

$$(\lambda_1 + \eta) Q_0 = \mu P_1 \tag{1}$$

$$(\lambda_1 + \eta) Q_n = \lambda_1 \sum_{k=1}^{n} Q_{n-k}\, g_k, \qquad n \geq 1 \tag{2}$$

$$\lambda_1 R_0 = \eta Q_0 \tag{3}$$

$$\lambda_1 R_n = \eta Q_n + \lambda_1 \sum_{k=1}^{n} R_{n-k}\, g_k, \qquad 1 \leq n \leq m-1 \tag{4}$$

$$(\lambda_1 + \gamma) D_m = \lambda_1 \sum_{k=1}^{m} R_{m-k}\, g_k + \eta Q_m \tag{5}$$

$$(\lambda_1 + \gamma) D_n = \lambda_1 \sum_{k=n-m+1}^{n} R_{n-k}\, g_k + \lambda_1 \sum_{k=1}^{n-m} D_{n-k}\, g_k + \eta Q_n, \quad n \geq m+1 \tag{6}$$

$$\lambda_1 U_m = \gamma D_m \tag{7}$$

$$\lambda_1 U_n = \gamma D_n + \lambda_1 \sum_{k=1}^{n-m} U_{n-k}\, g_k \quad m+1 \leq n \leq N-1 \tag{8}$$

$$(\lambda + \mu + \alpha) P_1 = \beta B_1 + \mu P_2 \tag{9}$$

$$(\lambda_2 + \mu + \alpha) P_n = \beta B_n + \mu P_{n+1} + \lambda_2 \sum_{k=1}^{n-1} P_{n-k}\, g_k, \qquad 2 \leq n \leq N-1 \tag{10}$$

$$(\lambda_2 + \mu + \alpha)\, P_n = \beta B_n + \mu P_{n+1} + \lambda_2 \sum_{k=1}^{n-1} P_{n-k}\, g_k + \gamma D_n + \lambda_1 \sum_{k=n-N+1}^{n-m} U_{n-k}\, g_k,\ n \geq N \qquad (11)$$

$$(\lambda_3 + \beta)\, B_1 = \alpha P_1 \qquad (12)$$

$$(\lambda_3 + \beta)\, B_n = \alpha P_n + \lambda_3 \sum_{k=1}^{n-1} B_{n-k}\, g_k, \qquad n \geq 2 \qquad (13)$$

IV PROBABILITY GENERATING FUNCTIONS(PGFs):

To obtain the system size distribution of the model, the following partial PGF's of R_n, D_n, U_n, P_n and B_n are defined.

$$R(z) = \sum_{n=0}^{m-1} R_n z^n; \qquad D(z) = \sum_{n=m}^{\infty} D_n z^n$$

$$U(z) = \sum_{n=m}^{N-1} U_n z^n; \qquad P_w(z) = \sum_{n=1}^{\infty} P_n z^n \qquad (14)$$

$$Q(z) = \sum_{n=0}^{\infty} Q_n z^n \text{ and } B(z) = \sum_{n=1}^{\infty} B_n z^n$$

For this we list the partial generating functions corresponding to different system state.

$$Q(z) = \mu P_1 \left(\frac{1 - V^*(w_X^1(z))}{w_X^1(z)} \right); \quad R(z) = \mu P_1 \psi(z)$$

$$D(z) = (\mu P_1 / \gamma)\, \left(D^*(w_X^1(z))\right) \left(V^*(w_X^1(z)) - w_X^1(z)\, \psi(z) \right), \quad U(z) = \mu P_1\, \phi^S(z)$$

where $w_X^1(z) = \lambda_1(1 - X(z))$

To calculate the total PGF $P^S{}_{(m,n)}(z)$, we first calculate the partial generating function of the system size when the server is idle($P_I(z)$) ; $P_I(z) = Q(z) + R(z) + D(z) + U(z)$

$$P_I(z) = \mu P_1\left\{\left(\frac{1-V^*(w_X^1(z))}{w_X^1(z)}\right)+\left(\frac{1-D^*(w_X^1(z))}{w_X^1(z)}\right)\left(V^*(w_X^1(z)) - w_X^1(z)\,\psi(z)\right)+\phi^s(z)+\psi(z)\right\}$$

(i.e) $P_I(z) = \mu P_1 I_S(z)$

where $I_S(z) = \left[\frac{1-V^*(w_X^1(z))\,D^*(w_X^1(z))}{w_X^1(z)} + D^*(w_X^1(z))\,\psi(z)+\phi^S(z)\right]$

$P_w(z) + B(z) = \left(1+(\alpha/\beta)\,B_r^*\,(w_X^3(z))\right)P_w(z)$

Thus the total PGF of the system size is given by;

$P^S_{(m,N)}(z) = P_I(z) + P_w(z) + B(z)$

$$P_w(z) = \frac{\left[-z\mu P_1 w_X^1(z)\right]\left[\left(\frac{1-D^*(w_X^1(z))\,V^*(w_X^1(z))}{w_X^1(z)}\right)+D^*(w_X^1(z))\,\psi(z)+\phi^s(z)\right]}{\left[\mu(z-1)+z\left(w_X^2(z)+\alpha\,(1-B_r^*(w_X^3(z)))\right)\right]} \tag{15}$$

$$B(z) = \frac{\alpha}{\beta}B_r^*(w_X^3(z))P_w(z)$$

The following identities are used to calculate the system size probabilities:

a. $\lim_{z\to 1}\left(\frac{1-V^*(w_X^1(z))}{w_X^1(z)}\right) = E(V)$

b. $\lim_{z\to 1}\left(\frac{1-V^*(w_X^1(z))D^*(w_X^1(z))}{w_X^1(z)}\right) = E(D)+E(V)$ (16)

c. $\lim_{z\to 1}\frac{-w_X^1(z)}{\left[\mu(z-1)+z\left(w_X^2(z)+\alpha\,(1-B_r^*(w_X^3(z)))\right)\right]} = \frac{\rho_1}{1-\rho_{23}^{br}}$

Where $\rho_{23}^{br} = \rho_2+\rho_3(\alpha/\beta)$ and $\rho_1 = \left(\frac{\lambda_1}{\mu}\right)E(X)$

V PERFORMANCE MEASURES

In this section, the expressions for the steady state probabilities are obtained. Let the steady state syatem size probabilities P_v, P_{build}, P_{setup}, P_{dor}, P_{busy} P_I and P_{Br} denote the probability that the server is in vacation, buildup, setup, dormant, busy and in breakdown state respectively.

$$P_V = (\mu P_1 / \eta)$$

$$P_{build} = \mu P_1 \sum_{n=0}^{m-1} \frac{\Psi_n}{\lambda_1}$$

$$P_{setup} = (\mu P_1 / \gamma)$$

$$P_{dor} = \mu P_1 \sum_{n=m}^{N-1} \frac{\phi_n^S}{\lambda_1}$$

$$P_I = P_V + P_{setup} + P_{dor} + P_{bulid}$$

$$P_I = \mu P_1 d_S (m, N)$$

where $d_S (m ,N) = 1/\eta + 1/\gamma + \sum_{n=0}^{m-1} \Psi_n + \sum_{n=m}^{N-1} \frac{\phi_n^S}{\lambda_1} = I_s(1)$

$$P_{busy} = \frac{\mu P_1 \rho_1}{1 - \rho_{23}^{br}} d_S(m,N) \tag{17}$$

and $P_{Br} = \alpha/\beta \, P_{busy}$

Thus the normalizing condition implies,

$P_{(m,N)}^S(1) = P_I + P_{busy} + P_{Br} = 1$

$$1 = \mu P_1 d_S(m,N) + \frac{\mu P_1 \rho_1}{1-\rho_{23}^{br}} d_S(m,N) + \alpha/\beta \, P_{busy}$$

$$= \mu P_1 d_S(m,N) \left[1 + \left(1 + \frac{\alpha}{\beta}\right) \frac{\rho_1}{1-\rho_{23}^{br}} \right]$$

$$= \mu P_1 d_S(m,N) \left[\frac{1-\rho_{23}^{br} + \rho_1^{br}}{1-\rho_{23}^{br}} \right]$$

i.e., $1 = \dfrac{\mu P_1 d_S(m,N)}{R}$

$$\mu P_1 = \frac{R}{d_S(m,N)} \tag{18}$$

where $R = \dfrac{1-\rho_{23}^{br}}{((1-\rho_{23}^{br})+\rho_1^{br})}$ with $\rho_i = (\lambda_i / \mu)\, EX$; $i = 1, 2, 3, \ldots$

$\rho_1^{br} = \rho_1 (1 + (\lambda / \beta))$ and $\rho_{23}^{br} = \psi_2 + \rho_3 (\lambda / \beta)$

Thus, $P_{busy} = \dfrac{\rho_1}{(1-(\rho_{23}^{br}-\rho_1^{br}))}$ (from equation 17)

By substituting for μP_1 from equation (18) the equation ($P^S_{(m,N)}(z)$) can be written as

$$P^S_{(m,N)}(z) = R\left[1-\frac{z\left(w_X^1(z)\left(1+(\alpha/\beta)\left(B_r^*\left(w_X^3(z)\right)\right)\right)\right)}{\mu(z-1)+z\left(w_X^2(z)+(\alpha/\beta)\ w_X^3(z)B_r^*\left(w_X^3(z)\right)\right)}\right]\frac{I_R(z)}{I_R(1)} \qquad (19)$$

where $I_S(z) = \left[\dfrac{1-V^*(w_X^1(z))\,D^*(w_X^1(z))}{w_X^1(z)} + D^*(w_X^1(z))\,\psi(z) + \phi^S(z)\right]$

VI DECOMPOSITION PROPERTY

Equation (19) implies that under the condition $\rho_{23}^{br} < 1$; the total PGF of the system size probabilities is the product of the PGF of two random variables one of which is

$$P_{M_i^X/M/1/BD}(z) = \frac{(1-\rho_{23}^{br})}{(1-(\rho_{23}^{br}-\rho_1^{br}))}\left[1-\frac{z\left(w_X^1(z)\left(1+\frac{\alpha}{\beta}\left(B_r^*\left(w_X^3(z)\right)\right)\right)\right)}{\mu(z-1)+z\left(w_X^2(z)+\frac{\alpha}{\beta}w_X^3(z)B_r^*\left(w_X^3(z)\right)\right)}\right]$$

This gives the PGF of the system size for the batch arrival $M_i^X/M/1/BD$ queueing model with heterogeneous arrival and unreliable server without (m,N) policy and without vacation and $\delta(z) = \dfrac{I_S(z)}{I_S(1)}$ gives the PGF of the conditional system size distribution during the server idle period (vacation + buildup + setup + dormant).

VII EXPECTED SYSTEM SIZE

In this section, the mean system sizes when the server is in different states are calculated. Let L_V, L_{build}, L_{setup}, L_{dor}, L_{busy} and L_{Br} denote the expected system size when the server is in vacation, buildup, setup, dormant, busy and breakdown state respectively.

To calculate the mean system size the following identities are used.

a. $$\frac{d}{dz}\left(\frac{1-V^*(w_X^1(z))}{w_X^1(z)}\right)_{z=1} = \lambda E(X)E\left(\frac{V^2}{2}\right)$$

b. $$\frac{d}{dz}\left(\frac{1-V^*(w_X^1(z))D^*(w_X^1(z))}{w_X^1(z)}\right)_{z=1} = \lambda_1 E(X)\left((ED^2/2)+ED\,EV+(EV^2/2)\right)$$

(20)

c. $$\frac{d}{dz}\left(\frac{-w_X^1(z)}{\left[\mu(z-1)+z\left(w_X^2(z)+\alpha\left(1-B_r^*(w_X^3(z))\right)\right)\right]}\right)_{z=1}$$

$$=\frac{\left(\lambda_1\mu E(X(X-1))+2\lambda_1 E(X)(\alpha\lambda_3 E(X))^2\,(E(B_r^2)/2)+\mu\rho_{23}^{br}\right)}{2\mu^2(1-\rho_{23}^{br})^2}$$

Then

$$L_V = \frac{d}{dz}(Q(z))_{z=1} = \mu P_1 \lambda_1 EX\,(E(V^2)/2)$$

$$L_{build} = \frac{d}{dz}(R(z)_{z=1} = \mu P_1 \sum_{n=0}^{m-1} n\,\psi_n\,(1/\lambda_1)$$

$$L_{setup} = \frac{d}{dz}(D(z))_{z=1} = \mu P_1 \lambda_1 E(X)\left((E(D^2)/2)+E(D)\left(E(V)+\psi(1)\right)\right)$$

$$L_{dor} = \frac{d}{dz}(U(z)_{z=1} = \mu P_1 \sum_{n=m}^{N-1} n\,\varphi_n^S\,(1/\lambda_1)$$

$$L_{Br} = \frac{d}{dz}(B(z))_{z=1} = \frac{\alpha}{\beta}\,[L_{busy}+\lambda_3\,(E(X)/\beta)P_{busy}] \quad \text{and}$$

$$L_{busy} = \frac{d}{dz}(P_w(z))_{z=1}$$

$$= \left\{ P_{busy} + \frac{\mu P_1 d_S(m,N)}{2(\mu(1-\rho_{23}^{br}))^2} \left[\lambda_1 \mu E(X)(X-1) \right.\right.$$

$$\left.\left. + 2\lambda_1 E(X)(\alpha \lambda_3 E(X)^2)(E(B_r^2)/2) + \mu \rho_{23}^{br} \right] + \mu P_1 \frac{\rho_1}{(1-\rho_{23}^{br})} l_S(m,N) \right\}$$

where $l_S(m, N) = l_0 + \lambda_1 E(X) E(D) \psi(1) + \sum_{n=0}^{m-1} \frac{n \psi_n}{\lambda_1} + \sum_{n=m}^{N-1} \frac{n \varphi_n}{\lambda_1}$ (21)

with $l_0 = \lambda_1 E(X)\left((E(D^2)/2) + E(D)E(V) + (E(V^2)/2)\right)$ (22)

Let $L^S(m,N)$ denote the expected system size of $M^X_{i(m,N)}/M/1/SV/BD$ queueing system under consideration. Then $L^S(m,N) = L_V + L_{build} + L_{setup} + L_{dor}$

$+ L_{busy} + L_{Br}$ Implies $L^S(m,N) = \dfrac{l_S(m,N)}{d_S(m,N)} + L_1$ (23)

where $L_1 = \dfrac{1}{(1-(\rho_{23}^{br} - \rho_1^{br}))} \left[\rho_1^{br} + \dfrac{\lambda_1 E(X(X-1))E(H) + \lambda_1\lambda_3(E(X^2)E(H^2))}{2(1-\rho_{23}^{br})} \right]$ (24)

with $E(H) = (1/\mu)(1 + (\alpha/\beta)) = ES(1 + (\alpha/\beta))$

and $E(H^2) = \alpha E Br^2 E(S)(1 + \rho_3 - \rho_2) + E(S^2)(1 + (\alpha/\beta))[(\lambda_2/\lambda_3) + (\alpha/\beta)]$

L_1 gives the mean system size of classical M_i^X/M/1/BD queueing model with unreliable server under restricted admissibility without vacation and without (m,N) policy.

VIII OTHER SYSTEM CHARACTERISTICS:

Busy period: The busy period begins, when the system size becomes at least N (soon after the set up work (or) at the end of the dormant period) and the server starts serving the customers and it ends, when the system next becomes empty and the server leaves for a vacation.

Let E(cycle), E(Busy) and E(I) denote the expected length of cycle, busy period and expected idle period. Then the long run fraction of time that the server is in set up state implies,

(i) $$P_{set} = \frac{E(D)}{E(cycle)} = \mu P_1 E(D), \quad \text{which gives} \quad E(cycle) = \frac{1}{\mu P_1}$$

Similarly,

(ii) $$\frac{E(Busy)}{E(cycle)} = P_{busy}$$ implies,

$$E(Busy) = P_{busy} E(Cycle) = d_s(m,N) \frac{\rho_1}{(1-\rho_{23}^{br})}$$ (from equation 17)

and

(iii) $$E(idle) = P_I E(cycle) = d_s(m,N) \qquad \text{follow by substituting for } P_I$$

Let $E(w_s)$ denote the expected waiting time in the system. Then the Little's formula implies

(iv) $$E(w_s) = \frac{L^s(m,N)}{\lambda_a E(X)}$$

where λ_a denote the actual arrival rate into the system which is given by,

$$\lambda_a = (\lambda r_1 P_I + \lambda r_2 P_{busy} + \lambda r_3 P_{Br}) = \lambda R(r_1 + (r_2 + r_3(\frac{\alpha}{\beta}))) \frac{\rho_1}{(1-\rho_{23}{}^{br})}$$

IX OPTIMAL MANAGEMENT POLICY:

In this section, the optimal values of m and N that minimize a linear cost function are obtained. To do this, the cost structure that has been widely used in the literature is employed (refer Yadin and Naor (1963), Ke (2003b), and Armuganathan and Jayakumar (2005)).

C_y (start up cost per cycle),C_{build} (cost per unit time for keeping the server idle (buildup)),C_h (holding cost per customer per unit time),C_{setup}(server set up cost per unit time for the preparatory work of the server before starting the service),C_{dor} (Server standby cost per unit time.),C_{busy}(Cost per unit time for keeping the server on and in operation.),C_V(Reward per unit time due to vacation.),C_{Br}(Breakdown cost per unit time for a failed server)

Let T_C^S(m, N) denote the total average cost per unit time for the system. Then

$$T_C^S(m, N) = \frac{C_y}{E(cycle)} + C_h L^S(m,N) + C_{build} P_{build} + C_{setup} P_{setup} + C_{dor} P_{dor}$$

$$+ C_{busy} P_{busy} - C_V P_V + C_{dor} P_{dor} \qquad (25)$$

Substituting for the system performances and rearranging the terms, T_C^S(m, N) Can be written as,

$$T_C^S(m, N) = \frac{1}{d_s(m,N)}\left[A^S + z_S(m) + C_{dor} R \sum_{n=m}^{N-1} \frac{\phi_n^S}{\lambda_1} + C_h \sum_{n=m}^{N-1} \frac{n\phi_n^S}{\lambda_1}\right] + A_1^s$$

where $A_1^S = (C_{busy} + \frac{\alpha}{\beta} C_{br}) P_{busy} + C_h L_1$, $A^s = R(C_y + C_{set} E(D) - C_V E(V) + C_h L_0$

and $Z_S(m) = (R\, C_{buid} + C_h \lambda_1 E(X) E(D)) \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda_1} + C_h \sum_{n=0}^{m-1} \frac{n\psi_n}{\lambda_1}$

and l_0, L_1 as in equations (22) and (24)

In order to find the optimal control values (m^*, N^*) that minimize T_C^S (m, N), a two dimensional search over the non-negative integer space must be made. Due to the mathematical complexity it is difficult to prove the convexity or unimodality of the cost functions T_C^S(m,N). This following the concept of the dynamic optimization (due to Bellman 1957, Ke 2001, 2003b and Lee and Park 1997) we consider the procedure that make it possible to calculate the optimal thresholds (m^*, N^*).

$$\text{Let } J(m, k) = \sum_{i=m}^{k} \phi_i \quad ; \quad M(m, k) = \sum_{i=m}^{k} i\,\phi_i^S$$

then $$T_C^S(m, k+1) - T_C^S(m, k) = \frac{\phi_k^S}{\lambda_1 d_S(m, k+1)\, d_S(m,k)} H_S(m,k) \qquad (26)$$

where $$H_S(m,k) = C_h \left(k\, l_m^S + \sum_{n=m}^{k-1} (k-n)\frac{\phi_n^S}{\lambda_1}\right) + R\, C_{dor}\, l_m^S - (A + Z_S(m)) \qquad (27)$$

with $l_m^S = E(D) + E(V) + \sum_{n=0}^{m-1} \frac{\psi_n}{\lambda_1}$

The equation (26) implies, for a given m, the sign of $H_S(m,k)$ determines whether T_C^S (m, k) increases (or) decreases w.r.t. k. Let n be the first k for which $H_S(m,k) > 0$.Then

$H_S(m,n+1) = H_S(m,n) + C_h \left(l_m^S + \sum_{i=m}^{k} \phi_i^S \right) > 0$ implies T_C^S(m, k) > T_C^S(m, n) for k > n.

This means that for a given m, the optimal value N^*(m) of N is given by the first k for which $H_S(m,k) > 0$ and that, once T_C^S(m, N(m)) increases with respect to N, it keeps on increasing thereafter. Therefore for a given m, T_C^S(m,N(m)) is conditionally unimodel and thereby N^*(m) conditionally optimal.

$$\text{Thus } N^*(m) = \min\{k \geq 1 / H_S(m,k) > 0\}, \tag{28}$$

where $H_S(m,k)$ is given as in equation (27)

Therefore for each m, T_C^S(m, N) has a relative minimum value at (m, N^*(m)). Thus the pair (m, N^*(m)) gives the relative optimal policy for a given m. Though it is difficult to prove mathematically that T_C^S(m, N) is convex or unimodular the computer experiments show that, the expected cost function is convex. Thus the optimal (m*,N^*) can be obtained by using the following algorithm.

CONCLUSION

A Bulk arrival, unreliable server with early setup and single vacation has been analyzed using P.G.F. This method works efficiently the steady state probabilities of the model considered. Further various performance measures and optimal management policy are derived and particular cases are also deduced. This is a part of the research work carried out to analyze breakdown queueing models with early setup and single vacation.

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