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Fixed point for generalized ξ-α expansive mappings in cone metric space

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Abstract

In this paper, first we introduce the notion of generalized (ξ, α) expansive maps for a pair of mappings in cone metric spaces and then prove some fixed point results for these maps.

Keywords. Cone metric space, fixed points, α-admissible maps, generalized (ξ, α) expansive mapping.

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1. INTRODUCTION

The study of fixed points of mappings satisfying certain contractive conditions has been a very active area of research. In 2007, L.G Huang and X. Zhang [8] introduced metric spaces by replacing the real numbers set with an ordered Banach space in co-domain. After that cone metric space was introduced which is generalization of metric space. Several authors worked on cone metric space and obtained many fixed and comman fixed point results. Several authors [13,11,19,1,15,12] worked on cone metric spaces and obtained many fixed and common fixed point results.
In the last decade, in [7] J. Gornickl and B.E Rhoades [14] used generalized contractive mapping to obtain common fixed point. Khan.et.al [10], Rhoades [14], Taniguchi [18], generalized the results for pair of mappings. Recently, Shahi.et.al [17] introduced the notion of (ξ-α) expansive map and proved fixed point results for such type of map.

In this paper we generalize that result in cone metric space and then introduced a new mapping of generalized (ξ-α) expansive mapping for a pair of maps.

2. PRELIMINARIES

First we need the following definitions and results that will be used subsequently (see [8]).
Let E be the real Banach space with a given norm $\|\,.\,\|_E$ and 0_E be the zero vector of E.

Definition 2.1.

Let E be the Real Banach space with a given norm $\|\,.\,\|_E$ and 0_E be the zero vector of E.
Then a non empty subset P of E is called a cone if and only if

(1) P is non-empty and $P \neq \{0\}$
(2) P is closed.
(3) $ax + by \in P$ for all $x, y \in P$ and $a, b \in R$ with $a, b \geq 0$ that is , P is convex.
(4) $P \cap (-P) = \{0_E\}$

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if
$y - x \in P$ .We write $x \leq y$ to indicate $x \leq y$ but $x \neq y$ and $x << y$ will stand for $y - x \in Int(P)$. (Int(P) = interior of P).

Definition 2.2.

Cone $P \subset E$ is called normal if there is a number K such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ where K is least positive number satisfying the above inequality and called normal constant of P.

Definition 2.3.

The cone $P \subset E$ is called regular if every increasing sequence which is bounded above is convergent. That is if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq x_3 \leq \ldots\ldots\ldots\ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x\| \to 0$ as $n \to \infty$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded below is convergent.

In the following, we suppose E is Banach space, P is cone in E with int $P \neq \phi$ and $\leq$ is a partial ordering with respect to P.

Definition 2.4.

Let X be non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies

(1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x = y$.

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(3) $d(x, y) \leq d(x, z) + (y, z)$ for all $x, y, z \in X$.

then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 2.5.

Let $E = R^2$, $P = \{(x, y) \in E \mid x, y \geq 0\} = R^2$, $X = R$ and $d : X \times X \to E$ such that $d(x, y) = (|x-y|, a|x-y|)$, where $a \geq 0$ is constant. Then (X, d) is a cone metric space.

Definition 2.6.

Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) Then $\{x_n\}$ is said to be convergent to x if every $c \in E$ with $0 << c$ there exist N such that
 $d(x_n, x) << c$ for all $n \geq N$.
 we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(2) If for every $c \in E$ with $0 << c$, there is a positive integer N such that for all n, m > N,
 $d(x_n, x_m) << c$. Then the sequence $\{x_n\}$ is called a Cauchy sequence in X.

(3) If every Cauchy sequence in X is convergent then (X, d) is called a complete cone metric space.

Lemma 2.7.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$

Lemma2.8.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$

Lemma2.9.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then limit of $\{x_n\}$ is unique. That is if $\{x_n\}$ is convergent to x and $\{x_n\}$ is convergent to y, then x = y.

Lemma2.10.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence converges to x, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma2.11.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X with $x_n \to x$ and $y_n \to y$ as $n \to \infty$.Then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

Recently Samet .et.al [15] introduced the notion of α – ψ contractive mappings and α-admissible mappings in metric space follows:

Definition2.12.

Let $T : X \to X$ and $\alpha : X\times X \to [0,\infty)$,we say that T is α-admissible if x, y ∈ X ,$\alpha(x, y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$.

Denote with Ψ the family of non –decreasing functions ψ: [0, +∞) → [0, +∞) such that
$\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each t > 0, where ψ^n is nth iteration of ψ.

Lemma2.13.([15]**)**

For every function ψ: [0, +∞) → [0, +∞) the following holds:
If ψ is non-decreasing, then for each t > 0, lim $\psi^n(t) = 0$ implies $\psi(t) < t$ and $\psi(0) = 0$.

Definition2.14.([15]**)**

Let (X, d) be a metric space and T: X→X be a mapping then T is said to be an α-ψ contractive mapping if there exist two functions $\alpha : X\times X \to [0,\infty)$ and ψ ϵ Ψ such that $\alpha(x, y)\, d(Tx, Ty) \leq \psi(d(x, y))$ for all x, y∈X

Shahi.et.al [17] introduce the new notion of (ξ, α) expansive mapping as follows:

Definition 2.15.

Let χ denote all the functions $\xi : [0,\infty) \to [0,\infty)$ which satisfy the following properties:-

(1) ξ is non decreasing.
(2) $\sum \xi^n(a) < \infty$ for each a > 0, where ξ^n is nth iteration of ξ.
(3) $\xi(a+b) = \xi(a) + \xi(b)$ for all a, b ∈ [0,∞)

Lemma 2.16 ([15]**)**

If ξ: $[0,\infty) \rightarrow [0,\infty)$ is a non decreasing function ,then for each a > 0, $\lim_{n\to\infty} \xi^n(a) = 0$ implies $\xi(a) < a$.

Definition 2.17([17]**)**

Let (X, d) be a metric space and T : X→X be a given mapping .we say that T is an (ξ, α) expansive mapping if there exist two functions $\xi \in \chi$ and $\alpha : X \times X \rightarrow [0,\infty)$ such that

$\xi(d(Tx, Ty)) \geq \alpha(x, y)\, d(x, y)$ for all x, y $\in$ X

Remark 2.18([17]**)**

If T: X→X is an expansive mapping ,then T is an (ξ, α) expansive mapping where $\alpha(x, y) = 1$ for all x, y $\in$ X and $\xi(a) = ka$ for all $a \geq 0$ and some $k \in [0,1)$

3. MAIN RESULT

Further Shahi.et.al[17] derived the following theorem in complete metric space.

Theorem 3.1.

Let (X, d) be a complete metric space and T:X→X be a bijective (ξ, α) expansive mapping satisfying the following conditions:

(i) $T^{-1}$is α-admissible.

(ii) There exist $x_0 \in X$ such that $\alpha(x_0, T^{-1}x_0) \geq 1$.

(iii) T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x$ as $n\rightarrow\infty$ then $\alpha(T^{-1}x_n, T^{-1}x) \geq 1$ for all n.

Then T has a fixed point, that is there exist $u \in X$ such that Tu = u.

Shahi.et.al[17] introduced the notion of f-α admissible of a map with respect to map g as follows:

Definition 3.2.

Let f, g : X → $[0,\infty)$ be mappings.The map f is α-admissible with respect to g if $\alpha(gx, gy) \geq 1$ implies $\alpha(fx, fy) \geq 1$for all x, y$\in$X (3.1)

Now we introduce the notion of generalized (ξ, α) expansive mapping for a pair of maps in cone metric space as follows:

Definition.3.3.

Let (X, d) be a cone metric space and P be a normal cone with normal constant K. Let f, g :X→X be mappings. We say that f and g are generalized (ξ, α) expansive mappings if there exist two functions $\xi \in X$ and $\alpha : X\times X\to[0,\infty)$ such that

$\xi(M(fx, fy)) \geq \alpha(gx, gy)d(gx, gy)$ for all x, $y\in X$ where

$$M(fx, fy) = \max\left\{ d(fx, fy)\, , \frac{d(fx,gx)+d(fy,gy)}{2}\, , \frac{d(fx,gy)+d(fy,gx)}{2}\right\} \tag{3.2}$$

Now we prove Theorem 3.1 for generalized (ξ, α) expansive mapping for a pair of linear maps in cone metric space:

Theorem 3.4.

Let (X, d) be a complete cone metric space .Let f, g : X→X be bijective generalized (ξ, α) expansive maps satisfying the following conditions:

(i) $f^{-1}$is α-admissible with respect to g^{-1}.

(ii) There exist $x_0 \in X$ such that $\alpha(g^{-1}x_0, f^{-1}x_0) \geq 1$

(iii) If $\{x_n\}$ be a sequence in X such that if $\alpha(g^{-1}x_n, g^{-1}x_{n+1}) \geq 1$ for all n implies $\alpha(x_n,x_{n+1}) \geq 1$.

(iv) If $\{gx_n\}$ be a sequence in X such that $\alpha(gx_n,gx_{n+1}) \geq 1$ for all n and if $gx_n \to gz \in g(X)$ as $n\to \infty$ then there exists, subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)},gx_n) \geq 1$ for all k.

Then f and g have a coincidence point.

Proof: In view of condition (ii), there exist $x_0\in X$ such that $\alpha(g^{-1}x_0, f^{-1}x_0) \geq 1$

Since f and g are bijective ,so we can choose $x_1\in X$ such that $g^{-1}x_1=f^{-1}x_0= y_0$

Continuing this process havingchoosenx_1, x_2,………x_n.we can choose x_{n+1} such that

$$g^{-1}x_{n+1}=f^{-1}x_n= y_n \tag{3.3}$$

That is $x_n = gy_{n-1} = fy_n$ for all $n \in N$. (3.4)

Now f^{-1} is α-admissible with respect to g^{-1}, we have $\alpha(g^{-1}x_0, f^{-1}x_0,) = \alpha(g^{-1}x_0, g^{-1}x_1) \geq 1$

Implies $\alpha(f^{-1}x_0, f^{-1}x_1) = \alpha(g^{-1}x_1, g^{-1}x_2) \geq 1$.continuing this process , we get

$\alpha(g^{-1}x_n, g^{-1}x_{n-1}) \geq 1$

in view of (iii) , we have $\alpha(x_n,x_{n+1}) \geq 1$ implies $\alpha(gy_{n-1},gy_n) \geq 1$.Now if $fy_{n+1} = fy_n$ for some n, then by (3.3) , f and g have a coincidence point at $y = y_n$. So assume $d(fy_n,fy_{n+1}) \geq 0$ for all n.

Consider $d(fy_n,fy_{n+1}) \leq \alpha(gy_{n-1}, gy_n)d(fy_n, fy_{n+1})$

$$\leq \alpha(gy_{n-1}, gy_n)d(gy_{n-1}, gy_n)$$

$$\leq \xi\, M(fy_{n-1}, fy_n) \tag{3.5}$$

Where $M(fy_{n-1}, fy_n) =$

$$\max\left\{d(fy_{n-1}, fy_n)\ \frac{d(fy_{n-1},gy_{n-1})+d(fy_n,gy_n)}{2}, \frac{d(fy_{n-1},gy_n)+d(fy_n,gy_{n-1})}{2}\right\}$$

$$= \max\left\{d\big(fy_{n-1}, fy_n\big) \frac{d(fy_{n-1},fy_n)+d(fy_n,fy_{n+1})}{2}, \frac{d(fy_{n-1},fy_{n+1})+d(fy_n,fy_n)}{2}\right\}$$

$$= \max\left\{d\big(fy_{n-1}, fy_n\big) \frac{d(fy_{n-1},fy_n)+d(fy_n,fy_{n+1})}{2}, \frac{d(fy_{n-1},fy_{n+1})}{2}\right\}$$

$$\leq \max\{d(fy_{n-1}, fy_n), d(fy_n, fy_{n+1})\}$$

Owing to monotonicity of function ξ and using inequalities (3.3) and (3.5) we get, for all $n \geq 1$,

$$d(fy_n, fy_{n+1}) \leq \xi(\max\{d(fy_{n-1}, fy_n), d(fy_n, fy_{n+1})\}) \tag{3.6}$$

If for some $n \geq 1$, we have $d(fy_{n-1}, fy_n) \leq d(fy_n, fy_{n+1})$ then from (3.6),we obtain

$d(fy_n, fy_{n+1}) \leq \xi\, d(fy_n, fy_{n+1})$.since P is normal cone with normal constant K, therefore $\|d(fy_n, fy_{n+1})\| \leq \|\xi(d(fy_n, fy_{n+1}))\| < \|d(fy_n, fy_{n+1})\|$

Which is contradiction. Thus for all $n \geq 1$, we have

$$\max\{d(fy_{n-1}, fy_n), d(fy_n, fy_{n+1})\} = d(fy_{n-1}, fy_n) \tag{3.7}$$

Now in view of (3.6) and (3.7) ,

$d(fy_n, fy_{n+1}) \leq \xi\, d(fy_{n-1}, fy_n)$

Continuing like this, we have

$d(fy_n, fy_{n+1}) \leq \xi^n d(fy_0, fy_1)$ for all n ≥ 1. (3.8)

Now for n ≥ m, using (3.8) and triangular inequality, we obtain-

$$d(fy_n, fy_m) \leq d(fy_n, fy_{n-1}) + d(fy_{n-1}, fy_{n-2}) + \ldots\ldots + d(fy_{m+1}, fy_m)$$

$$\leq \xi^{n-1} d(fy_0, fy_1) + \xi^{n-2} d(fy_0, fy_1) + \ldots\ldots + \xi^m d(fy_0, fy_1)$$

$$\leq (\xi^{n-1} + \xi^{n-2} + \cdots + \xi^m)\, d(fy_0, fy_1)$$

$$\leq \frac{\xi^m}{1-\xi} d(fy_0, fy_1)$$

Since P is normal cone with normal constant K, therefore - $\|d(fy_n, fy_m)\| \leq K \left\| \frac{\xi^m}{1-\xi} d(fy_0, fy_1) \right\|$

Which implies $d(fy_n, fy_m) \to 0$ as n, m → ∞. Hence $\{fy_n\}$ is Cauchy sequence in (X, d).

Since X is complete and g is bijective so there exist u∈X such that $gy_n = gu$ (3.9)

Now we show u is coincidence point of f and g.

On contrary assume d(fu, gu) ≥ 0,since by condition (iii) and (3.9) we have

$$d(fu, gu) \leq d(fu, gy_{n(k)}) + d(gy_{n(k)}, gu)$$

$$\leq d(fu, gy_{n(k)}) + \alpha(gy_{n(k)}, gu)\, d(gy_{n(k)}, gu)$$

$$\leq d(fu, gy_{n(k)}) + \xi(M((fy_{n(k)}, fu)) \quad (3.10)$$

Where $M((fy_{n(k)}, fu) = \max \left\{ d(fy_{n(k)}, fu), \frac{d(fy_{n(k)}, gy_{n(k)}) + d(fu, gu)}{2}, \frac{d(fy_{n(k)}, gu) + d(fu, gy_{n(k)})}{2} \right\}$

Owing to above inequality we get from (3.10)

$$d(fu, gu) \leq d(fu, gy_{n(k)}) + \xi\left(\max\left\{ d(fy_{n(k)}, fu), \frac{d(fy_{n(k)}, gy_{n(k)}) + d(fu, gu)}{2}, \frac{d(fy_{n(k)}, gu) + d(fu, gy_{n(k)})}{2} \right\}\right)$$

Since P is normal cone with normal constant K therefore,

$$\|\mathrm{d}(f\mathrm{u}, \mathrm{g}u)\|$$
$$\leq K\Bigg(\| \mathrm{d}(\mathrm{fu}, \mathrm{g}y_{n(k)})\|$$
$$+ \left\| \xi\left(\max\left\{\mathrm{d}(fy_{n(k)}, fu), \frac{d(fy_{n(k)}, \mathrm{g}y_{n(k)}) + d(fu, \mathrm{g}u)}{2}, \frac{d(fy_{n(k)}, \mathrm{g}u) + d(fu, \mathrm{g}y_{n(k)})}{2}\right\}\right)\right\|\Bigg)$$

Letting k → ∞ in above inequality,

$$\|\mathrm{d}(f\mathrm{u}, \mathrm{g}u)\| \leq \mathrm{K}\left\|\xi\left(\frac{d(fu,\mathrm{g}u)}{2}\right)\right\|$$
$$< \mathrm{K}\left\|\frac{d(fu,\mathrm{g}u)}{2}\right\|$$

Which is contradiction.Therefore $\|\mathrm{d}(f\mathrm{u}, \mathrm{g}u)\| \to 0$ as $n \to \infty$.This shows that f and g have coincidence point.

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