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Fuzzy Coloring Circular Interval Graphs of Integer Decomposition With Constant Variables

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Abstract

We provide a polynomial time combinatorial algorithm that computes the weighted coloring number and the corresponding colorings for fuzzy circular interval graphs. The algorithm reduces the problem to the case of circular interval graphs, then making use of a coloring algorithm .

We also show that the stable set polytopes of fuzzy circular interval graphs have the integer decomposition property.

Keywords: Interval graph, Weighted coloring, Decomposition property, Polytope STAB(G), Clav-free graphs, Fractional coloring.

INTRODUCTION

A weighted k coloringof a graph G with weights w: $D(G) \rightarrow \mathbb{N}_0$ is a multiset of stable set $S_1, \ldots S_k \subseteq D(G)$ such that each vertex $v \in D(G)$ is contained in w(v) many of these stable sets. The weighted coloring number $\chi_w(G)$ is the smallest k such that there exist stable sets as above. The problem of bounding and computing the weighted coloring number of graphs is a classical topic in combinatorics and graph theory and, for the class of quasi-line graphs and more specifically fuzzy circular interval graphs, has received a lot of attention recently.

From a polyhedral perspective, the weighted coloring problem has an interesting connection to the integer decomposition in the stable set polytopes of graphs. A polyhedron $P \subseteq R^n$ has the integer decomposition property, if each integer vector $z \in Z^n$ that is contained in $k \cdot P$ for some $k \in N$ can be decomposed into k integer vectors of P, i.e. there exist integer vectors $z_1, \ldots, z_k \in P$ such that

$$z = \sum_{i=1}^{k} z_i$$

The vectors $z_{1,} z_2, \ldots\ldots, z_k$ are called a k integer decomposition of z in P. There is a one to one correspondence between weighted colorings of a graph with weights w and integer decompositions of w in its stable set polytope. Moreover, if a stable set polytope has the integer decomposition property, and the maximum weighted stable set (MWSS) problem can be solved in polynomial time, then the weighted coloring number can be computed in polynomial time, via the equivalence of separation and optimization [6].

A graph is quasi-line if the neighborhood of each of its vertices is the union of two cliques. Chudnovsky and Seymor [2] provided a structural result that states that a connected quasi-line graph is a fuzzy circular interval graph or it is the composition of fuzzy linear interval strips with a collection of disjoint cliques. In particular line-graphs are quasi-line, and thus the weighted coloring problem for quasi-line graphs subsumes the NP-complete edge-coloring problem.

In this paper, we consider the subclass of fuzzy circular interval graphs and show that the weighted coloring problem can be solved in polynomial time. We will present two approaches to the problem: a purely combinatorial approach and a polyhedral approach based on linear programming. Both approaches work by reduction to circular interval graphs, exploiting their properties [12,5].

Our contribution. We present an efficient combinatorial algorithm to not only compute the coloring number, but also an optimal weighted coloring for fuzzy circular interval graphs. For a fuzzy circular interval graph G, it computes the weighted coloring number alone in time

$$o\left(|v(G)|^3 size(w)\right).$$

Given the coloring number, it computes an optimal weighted coloring in time

$$o\left(|v(G)|^5 + size(w)\right)$$

Here size(w) denotes the binary encoding length of w. The algorithm is based on a reduction to circular interval graphs using an algorithm for maximum b-matching and an algorithm of Gijswijt [5] to solve the weighted coloring problem on circular interval graphs. Our algorithm requires a so-called representation of the fuzzy circular interval graph as input. Such a representation can be computed in time $o\left(|v(G)|^3.|E(G)|\right) = o\left(|v(G)|^5\right)$; see [13].

We also show that the stable set polytopes of fuzzy circular interval graphs have the integer decomposition property, which leads to a linear programming based approach to compute the weighted coloring number.

The organization of this paper is as follows. In Section 1.1, we review important structural properties of fuzzy circular interval graphs that will be exploited by our algorithm. In Section 2, we present our combinatorial coloring algorithm. Finally in Section 3 we elaborate in more detail on the relation between weighted colorings and integer decompositions and prove the integer decomposition property for the stable set polytope of fuzzy circular interval graphs.

1.1 The structure of circular interval graphs

Given a graph G and a set of nodes S ⊆ D (G), a node d ∈ D (G) is said to be S-complete, if d is adjacent to every node of S. If d is adjacent to none of the nodes of S, d is said to be S-anticomplete. Given a node d ∈ V(G), we define the neighborhood of d as

$$N_G(d) := \{u \in D(G) : \{u, d\} \in E(G)\}.$$

Circular interval graphs are graphs G that can be obtained with the following construction. Let D (G) be a subset of a circle C. Further take a set I of intervals of the circle C. The set of edges E(G) is defined as follows. Two vertices are adjacent if and only if they are contained in a common interval of I.

The pair (D , I) completely describes a circular interval graph and is called a representation of G. These representations can be computed in linear time [3,9,7]. Fig. 1 shows an example for a circular interval graph.

Circular interval graphs can be colored efficiently. There is a combinatorial algorithm by Gijswijt [5] via integer decompositions for the stable set polytope of circular interval graphs. His result is the following.

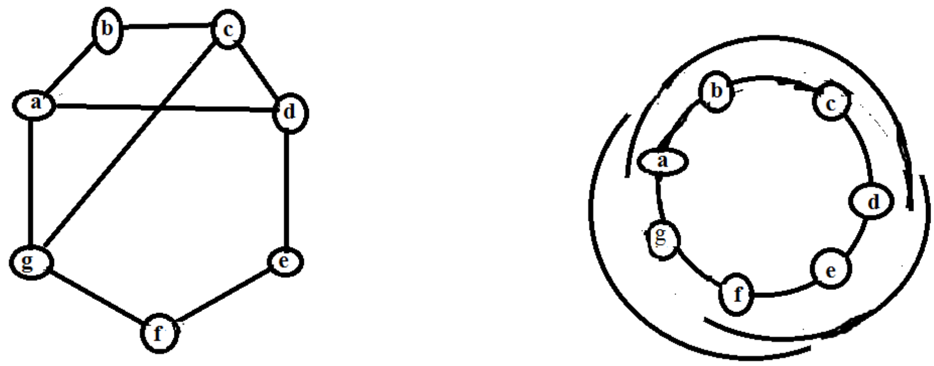

Fig. 1. A circular interval graph with its representation.

Theorem 1 ([5]). Given a circular interval graph G with weights w, for every d ∈ N we can decide if a weighted k coloring of (G, w) exists in time O (|S (G)|^3).

Proof

A weighted d coloring can be computed in time $o\left(|S(G)|^3 + size(w)\right)$. The number of different stable sets in the coloring is bounded by O (|S (G)|).

Fuzzy circular interval graphs [2] provide a generalization of the former class. They can be characterized as follows. A graph G is a fuzzy circular interval graph if there is a map Φ from S(G) to a circle C and a set I of intervals of C, none including another, such that no point of C is an endpoint of more than one interval so that:

- if three vertices a,b and c are adjacent, then $\phi(a), \phi(b)$ and $\phi(c)$ belong to a common interval;
- if two vertices a, b and c belong to the same interval, which is not an interval with endpoints $\phi(a), \phi(b)$ & $\phi(c)$, then they are adjacent.

For an interval [p, q,r] ∈ I, if both sets of preimages $A := \phi^{-1}(p), B := \phi^{-1}(q)$ and C:= $\phi^{-1}(r)$ are nonempty, (A, B)& (B,C) is called a fuzzy pair.

Note that by definition, both A ,B and C are cliques, but adjacencies between nodes of A,B & C can be arbitrary. Fuzzy pairs have another structural property that will be crucial later for our construction. Every node $a \in s(G) \setminus (A \cup B); b \in S(G) \setminus (B \cup C)$ is either A-complete or A-anticomplete. Similarly, a is either B-complete or B-anticomplete and , b is either C-complete or C-anticomplete

Analogous to circular interval graphs, the pair (ϕ, I) is called a representation of G. It completely defines all adjacencies, except for those of fuzzy pairs. Fig. 2 shows an example for a fuzzy circular interval graph and its representation. We remark that the definition of fuzzy pairs relies on the interval set I, and hence are dependent on a representation. Given two different representations of the same graph, the fuzzy pairs might differ. In the sequel when we speak of fuzzy pairs, we implicitly assume that a representation is given. Every fuzzy circular interval graph has a representation whose number of intervals is bounded by O (|S (G)|): the fact that no interval is allowed to include another limits the number of irredundant intervals. From now on we assume that the number of intervals is limited by O (|S (G)|). Representations for fuzzy circular interval graphs can be computed efficiently.

Theorem 2 ([13]). Given a graph G, one can decide whether G is a fuzzy circular interval graph and compute a suitable representation in time $o\left(|S(G)|^3 . |E(G)|\right)$.

Proof

The coloring algorithm presented later will reduce to the case of circular interval graphs to make use of Gijswijt's coloring algorithm. As fuzzy pairs are what distinguishes circular interval graphs from fuzzy circular interval graphs, they play an essential role in the transformation. A fuzzy pair (A, B) & (C,D) is called nontrivial if A∪B and C∪D contains an induced C4 subgraph, i.e. there are four nodes such that their

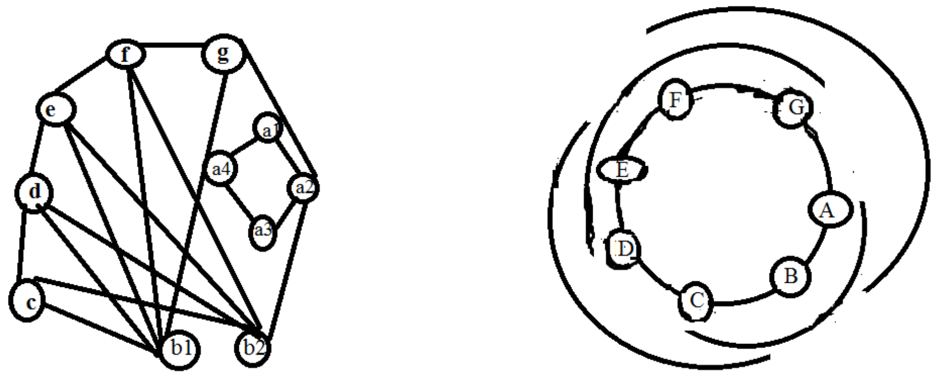

Fig. 2. A fuzzy circular interval graph with its representation. Here $\phi(a_1) = \phi(a_2) = \phi(a_3) = \phi(a_4) = A$, $\phi(b_1) = \phi(b_2) = B$, $\phi(c) = \phi(d) = D$, $\phi(e) = E$, $\phi(f) = F$ and $\phi(g) = G$

Induced subgraph is a cycle. It is called trivial otherwise. A crucial observation is that fuzzy circular interval graphs whose fuzzy pairs are all trivial are actually circular interval graphs; see, e.g. [4].

Lemma 1. Given a fuzzy circular interval graph G and a representation, if every fuzzy pair of G w.r.t. that representation is trivial, then G is a circular interval graph.

2. THE COLORING ALGORITHM

Our coloring algorithm for fuzzy circular interval graphs reduces to the case of circular interval graphs by transforming the input graph G^* and its weights w to a circular interval graph G^{**} with weights w^{**} such that the coloring number is preserved, i.e. $\chi w^*(G^*) = \chi w^{**}(G^{**})$. Then it applies Gijswijt's algorithm, see Theorem 1, to obtain a coloring of G^{**}, which finally is transformed to a coloring of G^*.

Lemma 1 suggests the following approach for the reduction of a fuzzy circular interval graph G: replace every nontrivial fuzzy pair in G with a trivial one in such a way that the weighted coloring number is preserved. This is done in several iterations, replacing the nontrivial fuzzy pairs one by one.

2.1. Fuzzy pair reduction

We now describe a single iteration, i.e. show how to replace a single fuzzy pair. Recall that fuzzy pairs (A, B) have the structural property that every node $d \notin A \cup B$ is either adjacent to all the nodes of A (of B) or to none of them. Thus as far as the stable sets of a coloring are concerned it is important to know whether a node of A (of B) is contained in a stable set whereas knowing the exact node itself is less important. Nodes in A and B can be re-distributed among the stable sets as long as they do not become adjacent in the sub-graph induced by $A \cup B$. This is reflected in the following construction to compact a fuzzy pair.

Consider a fuzzy circular interval graph G with weights w and a fuzzy pair (A, B) in G. Let $D^\circ := D(G)/(A \cup B)$. For a subset $S \subseteq D$we define $w(S) := \sum_{d \in s} w(d)$. The reduced graph (G′, w′) is defined as follows:

$D(G') := D^\circ \cup \{a_0, a_1, a_2, b_0, b_1, b_2\}$,

$$E(G') := E(G)|d^0 \cup \{\{d, a_0\}, \{d, a_1\}, \{d, a_2\} : d \in D^0 A - complete\}$$

$$\cup \{\{d, b_0\}, \{d, b_1\}, \{d, b_2\} : d \in D^0 B - complete\}$$

$$\cup \{\{a_0, a_1\}, \{a_0, a_2\}, \{a_1, a_2\}, \{b_0, b_1\}, \{b_0, b_2\}\}$$

$\{b_1,b_2\},\{a_0,b_0\},\{a_0,b_1\},\{a_0,b_2\},\{a_1,b_0\},$

$\{a_1,b_1\},\{a_1,b_2\},\{a_2,b_0\},\{a_2,b_1\},\{a_2,b_2\}\}.$

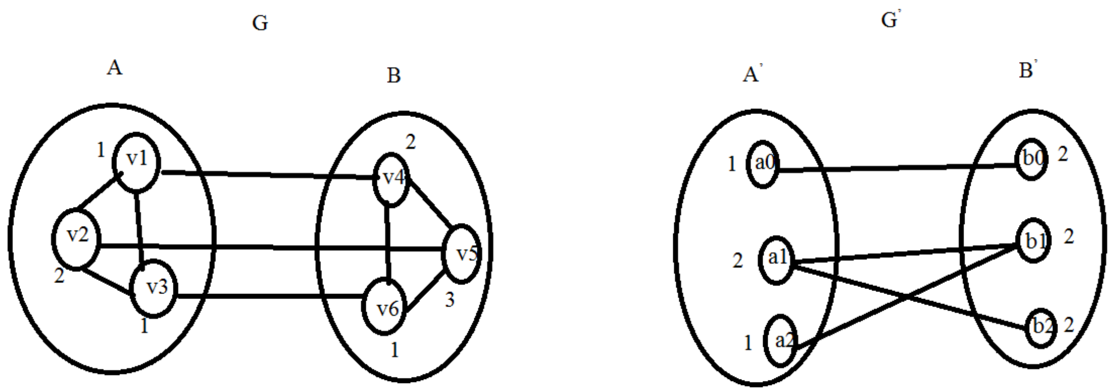

Fig. 3. A fuzzy pair (A, B) and its replacement (A′, B′) in the reduction. The numbers next to the nodes denote their weights.

$$w'(v) := \begin{cases} \alpha & \text{if } v=a_1 \text{ or } v=b_1 \\ w(A)-\alpha & \text{if } v=a_2 \\ w(B)-\alpha & \text{if } v=b_2 \\ w(v) & \text{else} \end{cases}$$

Notice that a similar construction is used in the independent work of Oriolo et al. [13] who designed an efficient recognition algorithm for fuzzy circular interval graphs.

We next specify α. The sets A and B together with the complement of the edges of G[A ∪ B] define a bipartite graph H. If a stable set S of G has two vertices in A ∪ B, then those two vertices are connected by an edge in H. Furthermore the set $(S \setminus (A \cup B)) \cup \{a_0, b_0\}$ is a stable set of G′. Writing a weighted k-coloring of G as the sum of characteristic vectors of stable sets $w = \chi S_1 + \ldots\ldots. + \chi S_k$, how many of the Si can contain two vertices of A ∪ B?

This number can be expressed as the size of a largest b-matching in H. Given node labels b: $D(H) \rightarrow N_0$, a b-matching is a multiset of edges of E(H) such that each node d ∈ D (H) is covered by at most b(d) of those edges. Alternatively one can define a maximum b-matching as an optimal solution to the linear program

$$\max\left\{ \sum_{e\in E(H)} x_e : \sum_{e\in\delta(v)} x_e \le b(d)\forall d \in D(H); x_e \ge 0 \forall e \in E(H) \right\} \text{--------------(1)}$$

Since H is bipartite, the vertices of the linear program (1) are integral; see, e.g. [15].

Now setting b := w, the number of stable sets Si that can contain two vertices of A ∪ B is clearly bounded by the size of a largest b-matching, as a coloring with ℓ many of those stable sets directly gives rise to a b-matching of size ℓ. The number α from the reduction above is the size of a largest b-matching or equivalently the optimum value of the linear program (1). We remark that this number can be computed efficiently using a combinatorial max s − t-flow algorithm, e.g. Karzanov's preflow push algorithm [15]. Fig. 3 illustrates an example for the reduction.

In order for the reduced graph to be useful for our reduction, we need to prove that it satisfies the following three properties.

- The reduction preserves the structure of the graph, i.e. G′ is still a fuzzy circular interval graph.
- If (A, B) was nontrivial, the number of nontrivial fuzzy pairs has been reduced by one.
- We have $\chi_w(G) = \chi_{w'}(G')$.

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