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Strong Convergence Theorem for Common Solution of Variational Inequality and Fixed Point of λ-Strictly Pseudo-contractive Mapping in Uniformly Smooth Banach Space

Poonam Mishra [1]
Department of Applied Mathematics,
Amity School of Engineering and Technology,
Amity University, Raipur (C.G.) 492001, India.

Raksha Rani Agrawal
Department of Applied Mathematics,
Shri Shankaracharya Technical Campus,
Bhilai (C.G.), India.

Abstract

In this paper,strong convergence of Modified Mann iterative algorithm is proved for λ-strictly pseudocontractive mapping in a uniformly smooth real Banach space. It is also a solution of a certain variational inequality. Our result extends the result of Marino et al. [17] and other such results.

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1. Introduction

Let E be a real Banach space and E^* its dual space. Let us denote by $J_q (q > 1)$ the generalized duality mapping from E into 2^{E^*}, that is, $j_q : E \to 2^{E^*}$ which is defined as

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = ||x||^q and ||f|| = ||x||^{q-1}\},$$

[1]Corresponding Author: ppoonam22@gmail.com

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between E and E*. In particular, $J_q = J_2$ is called the normalized duality mapping and $J_q(x) = ||x||^{q-2} J_2(x)$ for $x \neq 0$. If $E := H$ is a real Hilbert space, then $J = I$ where I is the identity mapping. It is well known that if E is smooth, then J_q is single-valued , which is denoted by j_q [25].

Let $U = \{x \in X : ||x|| = 1\}$.A Banach space E is said to be strictly convex if $\dfrac{||x+y||}{2} \leq 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. It is also called uniformly convex if $\lim ||x_n - y_n|| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $||x_n|| = ||y_n|| = 1$ and $\lim \dfrac{\|x_n + y_n\|}{\|2\|} = 1$.

A Banach space E is said to be Gateaux differentiable if the limit $\lim\limits_{t \to 0} \dfrac{||x+ty|| - ||x||}{t}$ exists for all $x, y \in U$. In this case E is smooth. Also , we define a function $\rho_X : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of E as follows:

$$\rho_X(t) = \sup\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : x \in U, ||y|| < t\}$$

A Banach space E is said to be uniformly smooth if $\dfrac{\rho_x(t)}{t} \to 0$ as $t \to 0$.

Definition 1.1. A mapping $T : C \to C$ is said to be λ-strictly pseudocontractive [4] , if for all $x, y \in C$ there exist $\lambda > 0$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \leq ||x-y||^q - \lambda||(I-T)x - (I-T)y||^q,$$

or, equivalently

$$\langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq \lambda||(I-T)x - (I-T)y||^q.$$

The convergence of iterative sequences for strictly pseudocontractive mapping were initiated by Browder and Petryshyn [4] in 1967 while studying the nature of non-expansive mapping. However later, it was found that the applications of strictly -pseudocontractive mappings are vast when it is associated with inverse strongly monotone operators. Consequently studies were made of iterative sequences that converges strongly to fixed point of a strictly pseudocontractive mapping and relating them to the solution of a certain variational inequality.The reason was the classical Variational Inequality problem which was first introduced by Stampachhia [24] in 1964. Since then many researchers have studied wide class of unrelated problems in the unified and general framework of variational theory. In fact, nonlinear equilibrium, dynamical network, optimal design, bifurcation and chaos, and moving boundary problems arising in various branches of pure and applied sciences can be studied via variational inequalities.

The problem of finding fixed points for λ-strictly pseudocontractive mappings using the Mann method [16] in suitable space structure has been studied by many authors. Mann iteration [21] was first used to obtain fixed points of non expansive mapping given by an iterative sequence $\{x_n\}_{n=1}^{\infty}$ for $x_1 \in K$ where K is a convex subset of a Banach Space E and $T : K \to K$ is a nonexpansive mapping, such that

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \tag{1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0, 1] satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

It was found that in an infinite-dimensional Hilbert space, the Mann iteration (1) converges weakly for λ- strictly pseudocontractive mapping, and even in general for nonexpansive mappings. It therefore implied that Mann iteration (1) has to be modified in order to prove its strong convergence.

In 2007, Marino and Xu [17] generated an iterative sequence using Mann algorithm (1) and proved weak convergence for k-strictly pseudo-contractive mapping in Hilbert Space. But then they used CQ algorithm to obtain strong convergence for finite family of λ-strictly pseudocontractive mapping in Hilbert space.

Also, Acedo and Xu [1] using Mann iteration obtained weak convergence for finite family of λ-strictly pseudocontractive mapping in Hilbert space and then used CQ algorithm to obtain strong convergence for finite family of λ-strictly pseudocontractive mapping.

Zhou [30], in 2008, proved the weak convergence for λ-strictly pseudocontractive mapping in real 2-uniformly smooth Banach space and then made a modification in Mann iteration to obtain strong convergence for λ-strictly pseudocontractive mapping.

Zhang and Su [29] in 2009, extended the results of Marino and Xu [17], first obtained weak convergence results using Mann iteration (1) for λ-strictly pseudocontractive mappings in real q-uniformly smooth Banach space and further obtained strong convergence for the finite family of λ-strictly pseudocontractive mapping in q-uniformly smooth Banach space using a modification of Mann iteration method.

In same year 2009, Zhang and Guo [28] using Mann iteration (1) established the weak convergence for λ-strictly pseudocontractive mappings in a real q-uniformly smooth and uniformly convex Banach space which was an improvement of the result of Osilike et al. [19]. They in fact proved the following theorem:

Theorem 1.2. [28] Let E be a real q-uniformly smooth and uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $T : K \to K$ be a λ-strictly pseudocontractive mapping with $F(T) \neq \phi$ Let $\{a_n\}$ be a real sequence satisfying the condition:

$$\mu \leq \alpha_n < 1$$

$$\sum_{n=0}^{\infty}(1 - \alpha_n)[q\lambda - C_q(1 - \alpha_n)_{q-1}] = \infty$$

where

$$\mu \in \left[\max\left\{\theta, 1-\left(\frac{q\lambda}{C_q}\right)_{\frac{1}{q-1}}\right\}, 1\right].$$

Given $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Mann's algorithm (1). Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.

In the above result, obviously the Mann iteration needs to be modified and the conditions upon λ-strictly pseudocontractive mapping needs to be strengthen in order to establish the strong convergence of the iterative sequence.

Apart from using Modified Mann Method, Katchang et al. [13] used Modified Ishikawa Iterative method to obtain strong convergence for an infinite family of strictly pseudocontractive mappings in Banach spaces given by:

Theorem 1.3. **[13]** Let E be a real q-uniformly smooth and strictly convex Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Let C be a nonempty closed and convex subset of E which is also a sunny nonexpansive retraction of E such that $C+C \subset C$. Let A be a strongly positive linear bounded operator on E with coefficient $\bar{\gamma} > 0$ such that $0 < \bar{\gamma} < \bar{\gamma}/\alpha$, and let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Let S_i, $i = 1, 2, ...$, be λ_i-strictly pseudocontractive mappings from C into itself such that $\cap_{n=1}^{\infty} F(S_n) \neq \phi$ and $\inf\limits_{\lambda_i} > 0$. Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in (0, 1) satisfy the following conditions:

(i) $\sum\limits_{n=0}^{\infty} a_n = \infty;\, and \lim\limits_{n\to\infty} a_n = 0,$

(ii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1,$

(iii) $\lim\limits_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0,$

(iv) $\delta_n(1+\gamma_n) - 2\gamma_n > a$, for some a $\in (0, 1)$

and the sequence $\{\theta_{n,k}\}$ satisfies some Hypothesis. Then, the sequence $\{x_n\}$ generated by $x_0 \in C$ chosen arbitrarily,

$$z_n = \delta_n x_n + (1-\delta_n) W_n x_n$$

$$y_n = \gamma_n x_n + (1-\gamma_n) W_n z_n$$

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A) y_n,\ \forall n \geq 0$$

converges strongly to $x^* \in \cap_{n=1}^{\infty} F(S_n)$, which solves the following variational inequality:

$$\langle \gamma f(x^*) - Ax^*, J(p-x^*)\rangle \leq 0,\ \forall f \in \Pi_C,\ p \in \cap_{n=1}^{\infty} F(S_n)$$

Another method which was used to obtain strong convergence for λ-strictly pseudocontractive mapping was given by Yao et al. [27]. They used Halpern type iterative method to obtain strong convergence for λ strictly pseudo-contractive mapping in a q-uniformly smooth Banach space. They gave following result:

Theorem 1.4. [27] Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space E. Let $f : C \to C$ be a ρ-contraction. Let $T : C \to C$ be a λ strictly pseudo-contractive mapping such that $F(T) \neq \phi$. For $t \in (0, 1)$, defined a net $\{x_t\}$ by $x_t = tf(x_t) + (1-t)Tx_t$. Then, as $t \to 0$, the net $\{x_t\}$ converges strongly to $p \in F(T)$ which solves the following variational inequality

$$\langle (I - f)p, j(x - p)\rangle \geq 0; \quad forall x \in F(T)$$

Recently, Cho and Kang [7] considered the convex feasibility problem and constructed a hybrid iterative method based on Mann and relaxed extragradient method to approximate strong convergence for common element of solution set of classical variational inequality and fixed point set of strictly pseudocontractive mapping in framework of Hilbert spaces.

Further, Nazari et al. [18] improved and extended the result of Cho and Kang [7] from Hilbert space to a more general q-uniformly smooth Banach space using a modification of hybrid [7] iterative sequence $\{x_n\}$ for family of λ-strictly pseudo-contractive mappings $\{T_n\}$ with $0 < \lambda < 1$ as

$$\begin{cases} x_1 \in C \\ y_n = \alpha_n x_n + (1-\alpha_n) \sum_{m=1}^{r} \eta_n^m Q_c(x_n - \lambda_m A_m x_n) \\ x_{n+1} = Q_c[\beta_n \gamma f x_n + (I - \beta_n \mu F) S_n y_n], \ n \geq 1, \end{cases}$$

for the sequences in (0, 1).Then x_n converges strongly to $x^* \in Fix(T_n)$ as $n \to \infty$ in which x^* is unique solution of Variational inequality

$$\langle (\mu F - \gamma f)x^*, j_q(x^* - p)\rangle \leq 0 \ \ \forall p \in F(S)$$

In 2015, Yekini Shehu [23] using another Modified Mann method proved the strong convergence of an iterative sequence to approximate the fixed point of λ-strictly pseudocontractive mapping in a uniformly smooth real Banach space. It extended in fact the results of Li and Yao [14].

In the same year another important modification of Mann iteration method to obtain strong convergence using non-expansive mapping was given by Hussain et al. [10] who used the following iterative method:

Theorem 1.5. [10] Let H be a Hilbert space and $T : H \to H$ a non-expansive mapping. Let (α_n), (μ_n) be sequences in (0,1] such that

- $\lim_{n} \to \infty = 0;$

- $\sum_{n=1}^{+\infty} \alpha_n \mu_n = +\infty$;
- $|\mu_{n+1} - \mu_n| = o(\mu_n)$;
- $|\alpha_{n+1} - \alpha_n| = o(\alpha_n \mu_n)$

then the sequence (x_n) generated by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n - \alpha_n \mu_n x_n$$

strongly converges to a point $x^* \in Fix(T)$ with minimum norm

$$\|x^*\| = \min_{x \in Fix(T)} ||x||$$

Last year, Marino et al. [17] extended the above [10] modified Mann method to find strong convergence of fixed point of k-strictly pseudocontractive mapping in Hilbert space which was also a solution of certain variational inequality. They observed that above modification was closest to original Mann as compare to other modifications. They proved:

Theorem 1.6. [17] Let H be a Hilbert space and let C be a nonempty closed cone of H. Let $T : C \to C$ be a k-strictly pseudo-contractive mapping such that $Fix(T) \neq \phi$. Suppose that $(\alpha_n)_{n \in \mathbb{N}}$ and $(\mu_n)_{n \in \mathbb{N}}$ are real sequences, respectively, in $(k, 1)$ and in (0,1) satisfying the conditions:

(1) $k < \liminf_{n \to \infty} \leq \limsup_{n \to \infty} \alpha_n < 1$

(2) $\lim_{n \to \infty} \mu_n = 0$

(3) $\sum_{n=1}^{\infty} \mu_n = \infty$

A sequence $(x_n)_{n \in \mathbb{N}}$ is defined as:

$$x_1 \in C, x_{n+1} = \alpha_n (1 - \mu_n) x_n + (1 - \alpha_n) T x_n, \ n \in \mathbb{N}$$

Then $(x_n)_{n \in \mathbb{N}}$ converges strongly to $\bar{x} \in Fix(T)$ that is, the unique solution of the variational inequality

$$\langle -\bar{x}, y - \bar{x} \rangle \leq 0, \quad \forall y \in Fix(T)$$

Now, the question is:
Is this possible to prove the above strong convergence of λ strictly pseudocontractive mapping in Banach space also? The above question, we reply affirmatively in this paper.

Inspired by above results [18], [17], [23], [7], [10], we have proved in this paper, the strong convergence of a Modified Mann iterative algorithm to a fixed point of a λ-strictly pseudocontractive mapping in a uniformly smooth real Banach space and also shown that it is a solution of a certain variational inequality.

2. Preliminaries

In order to prove our main result we the following Lemmas:

Lemma 2.1. [26] Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where

- $(\alpha_n)_{n\in\mathbb{N}} \subset [0,1], \sum_{n=1}^{\infty} \alpha_n = \infty;$
- $\limsup_{n\to\infty} \sigma_n \leq 0;$
- $\gamma_n \geq 0, \sum_{n=1}^{\infty} \gamma_n < \infty$

Then we have

$$\lim_{n\to\infty} a_n = 0$$

Lemma 2.2. [15] Let $(\gamma_n)_n \in \mathbb{N}$ be a sequence of real numbers such that there exists a subsequence $(\gamma_{n_j})_{j\in\mathbb{N}}$ of $(\gamma_n)_{n\in\mathbb{N}}$ such that $\gamma_{n_j} < \gamma_{n_j+1}$,for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $(m_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large)numbers $k \in \mathbb{N}$:

$$\gamma_{m_k} \leq \gamma_{m_k+1} \; and \; \gamma_k \leq \gamma_{m_k+1}.$$

In fact, m_k is the largest number n in the set $1, \cdots, k$ such that the condition $\gamma_n \leq \gamma_{n+1}$ holds.

A Banach space X is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in X which converges weakly to x, then

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n} ||x_n - y|| \;\; \forall y \in X, \; y \neq x$$

By [[9] Theorem 1], we know that if X admits a weak sequentially continuous duality mapping, then x satisfies Opial's condition.The following Lemma will be needed in the sequel:

Lemma 2.3. [2, 11, 25] Let C be a nonempty closed convex subset of a reflexive Banach Space X that satisfies Opial's condition and suppose that $T : C \to C$ is nonexpansive then mapping $I - T$ is demiclosed at zero, i.e. if $x_n \to x$ and $||x_n - Tx_n|| \to 0$ then $Tx = x$.

One of the important lemma in this regard as given by Marino et al. [17] is

Lemma 2.4. [17] Let C be a non-empty,closed, and convex subspace of H, T a mapping from C into itsef such that I-T is demiclosed at 0, let $(y_n) \subset C$ be a bounded sequence. If $||y_n - Ty_n|| \to 0$, then

$$\limsup n\langle -\bar{p}, x - \bar{p}\rangle \leq 0,$$

where $\bar{p} = P_{Fix(T)}(0)$ is the unique point in Fix(T) that satisfies the variational Inequality

$$\langle -\bar{p}, x - \bar{p}\rangle \leq 0, \quad \forall x \in Fix(T). \tag{2}$$

Lemma 2.5. [6] Let X be a Banach Space and J be a normalized duality mapping. Then for any given $x, y \in X$, the following inequality holds:

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y)\rangle, \quad \text{for all } \ j(x + y) \in J(x + y).$$

3. Main Result

Now we prove our main result as follows:

Theorem 3.1. Let K be a nonempty closed convex subset of a uniformly smooth real Banach space E and Let $T : K \to K$ be a λ-strictly pseudo–contractive mapping such that $Fix(T) \neq \phi$.Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ are real sequences, respectively, in $(\lambda, 1)$ and in $(0, 1)$ satisfying the conditions:

(1) $\lambda < \liminf\limits_{n\to\infty} \alpha_n \leq \limsup\limits_{n\to\infty} \alpha_n < 1$;

(2) $\lim\limits_{n\to\infty} \mu_n = 0$;

(3) $\Sigma_{n=1}^{\infty}\mu_n = \infty$.

Let us define a sequence $(x_n)_{n\epsilon\mathbb{N}}$ as follows:

$$x_1 \in K, x_{n+1} = \alpha_n(1 - \mu_n)x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N} \tag{3}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges strongly to $\bar{x} \in Fix(T)$, that is, the unique solution of the Variational Inequality

$$\langle -\bar{x}, y - \bar{x}\rangle \leq 0, \quad \forall y \in Fix(T)$$

Proof. We shall first show that $(x_n)_{n\in(N)}$ is bounded. We observe from the conditions $\mu_n \to 0$ and $k < \liminf \alpha_n \leq \limsup \alpha_n < 1$, it follows that there exists an integer $n_0 \in \mathbb{N}$ such that

$$\mu_n \leq 1 - \frac{k}{\alpha_n}, \quad \forall n \geq n_0$$

that is,

$$k - \alpha_n(1 - \mu_n) \leq 0 \tag{4}$$

Let we have $p \in Fix(T)$ and put $r = \max\{||x_{n0} - p||, ||p||\}$. Then we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n[(1-\mu_n)(x_n - p) + \mu_n(-p)] + (1-\alpha_n)[Tx_{n-p}]\| \\
&\leq \|\alpha_n(1-\mu_n)(x_n - p) + (1-\alpha_n)(Tx_n - p)\| + \mu_n\|p\| \\
&= \|\alpha_n(1-\mu_n)(x_n - p) - (1-\alpha_n)(x_n - tx_n)\| + \mu_n\|p\|
\end{aligned} \tag{5}$$

Now, from (5) and Lemma 2.1

$$\begin{aligned}
&\|\alpha_n(1-\mu_n)(x_n - p) - (1-\alpha_n)(x_n - Tx_n)\|^2 \\
&\leq \alpha_n^2(1-\mu_n)^2\|x_n - p\|^2 + (1-\alpha_n)^2 c\|x_n - Tx_n\|^2 \\
&\quad - 2\alpha_n(1-\alpha_n)(1-\mu_n)\langle x_n - Tx_n, j(x_n - p)\rangle \\
&= \alpha_n^2(1-\mu_n)^2\|x_n - p\|^2 + (1-\alpha_n)^2 c\|x_n - Tx_n\|^2 \\
&\quad - 2\lambda\alpha_n(1-\mu_n)(1-\alpha_n)\|x_n - Tx_n\|^2 \\
&= \alpha_n^2(1-\mu_n)^2\|x_n - p\|^2 - (1-\alpha_n)(2\lambda\alpha_n(1-\mu_n) \\
&\quad - c(1-\alpha_n))\|x_n - Tx_n\|^2 \\
&\leq (1-\mu_n)^2\|x_n - p\|^2
\end{aligned} \tag{6}$$

from (5) and (6), it follows

$$\begin{aligned}
\|x_{n+1} - p\| \leq (1-\mu_n)\|x_n - p\| + \mu_n\|p\| \\
&\leq max\{\|x_n - p\|, |\|p\|\} \\
&\ \ \vdots \\
&\leq max\{\|x_n - p\|, \|p\|\}
\end{aligned}$$

Hence x_n is bounded and also is Tx_n.
Now we shall prove that for $p \in Fix(T)$

$$(1-\alpha_n)(\alpha_n - \lambda)||x_n - Tx_n||^2 \leq (||x_n - p)^2 - ||x_{n+1} - p||^2) - 2\alpha_n\mu_n\langle x_n, j(x_{n+1} - p)\rangle \tag{7}$$

we have

$$\begin{aligned}
x_{n+1} - p &= \alpha_n(1-\mu_n)x_n + (1-\alpha_n)Tx_n - p \\
&= (x_n - p) - (1-\alpha_n)(x_n - Tx_n - \alpha_n\mu_n x_n
\end{aligned}$$

and so

$$\begin{aligned}
||x_{n+1} - p||^2 &\le ||(x_n - p) - (1 - \alpha_n)(xn - Tx_n)||^2 - 2\alpha_n\mu_n\langle x_n, j(x_{n+1} - p)\rangle \\
&= ||x_n - p||^2 - 2(1 - \alpha_n)\langle x_n - Tx_n, j(x_n - p)\rangle \\
&\quad + c(1 - \alpha_n)^2||x_n - Tx_n||^2 - 2\alpha_n\mu_n\langle x_n, j(x_{n+1} - p)\rangle \\
&\le ||x_n - p||^2 - 2(1 - \alpha_n)\lambda||x_n - Tx_n||^2 + c(1 - \alpha_n)^2||x_n - Tx_n||^2 \\
&\quad - 2\alpha_n\mu_n\langle x_n, j(x_{n+1} - p)\rangle \\
&\le ||x_n - p||^2 - (1 - \alpha_n)(2\lambda - 1 + \alpha_n)||x_n - Tx_n||^2 \\
&\quad - 2\alpha_n\mu_n\langle x_n, j(x_{n+1} - p)\rangle \\
&\le ||x_n - p||^2 + (1 - \alpha_n)(\lambda_n - \alpha_n)||x_n - Tx_n||^2 \\
&\quad - 2\alpha_n\mu_n\langle x_n, j(x_{n+1} - p)\rangle
\end{aligned}$$

So, (7) is proved.
Moreover, since $\alpha_n \in (\lambda, 1)$.

$$||x_{n+1} - p||^2 \le ||x_n - p||^2 - 2\alpha_n\mu_n\langle x_n, j(x_{n+1} - p)\rangle$$

Now we prove strong convergence of (x_n) concerning two cases:

Case I
Suppose that $||x_n - p||$ is monotone nonincreasing. Then $||x_n - p||$ converges and hence

$$\lim_{n\to\infty} ||x_{n+1} - p||^2 - ||x_n - p||^2 = 0$$

From this and from the assumptions

$$\lim_n \mu_n = 0, and \lambda < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$$

by (7) we get

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0$$

from this and boundedness of (x_n) , thanks to demiclosedness of I-T we deduce

$$\omega_\ell(x_n) \subseteq Fix(T).$$

Now we put

$$\begin{aligned}
z_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n \\
&= (1 - (1 - \alpha_n))x_n + (1 - \alpha_n)Tx_n \\
&= x_n - (1 - \alpha_n)x_n + (1 - \alpha_n)Txn
\end{aligned}$$

from which we have

$$z_n - x_n = (1 - \alpha_n)(Tx_n - x_n) \tag{8}$$

Hence we find that

$$\begin{aligned} x_{n+1} &= \alpha_n(1-\mu_n)x_n + (1-\alpha_n)Tx_n \\ &= (1-\alpha_n)(Tx_n - x_n) + x_n - \alpha_n\mu_n x_n \\ &= z_n - x_n + x_n - \alpha_n\mu_n x_n \qquad from(6) \\ x_{n+1} &= z_n - \alpha_n\mu_n x_n \\ &= z_n - \alpha_n\mu_n x_n - z_n\alpha_n\mu_n + z_n\alpha_n\mu_n \\ &= (1-\alpha_n\mu_n)z_n + \alpha_n\mu_n(z_n - x_n) \qquad from(6) \\ &= (1-\alpha_n\mu_n)z_n + \alpha_n\mu_n(1-\alpha_n)(Tx_n - x_n) \end{aligned} \tag{9}$$

Let $\bar{x} = P_{Fix(T)}(0) \in Fix(T)$ the unique solution of the variational inequality

$$\langle -\bar{x}, j(y - \bar{x})\rangle \leq 0, \forall y \in Fix(T) \tag{10}$$

From definition of z_n

$$\begin{aligned} ||z_n - x_n||^2 &= ||x_n - \bar{x} - (1-\alpha_n)(x_n - Tx_n)||^2 \\ &= ||x_n - \bar{x}||^2 - 2(1-\alpha_n)\langle x_n - Tx_n, j(x_n - \bar{x})\rangle + (1-\alpha_n)||x_n - Tx_n||^2 \\ &\leq ||x_n - \bar{x}||^2 - 2\lambda(1-\alpha_n)||x_n - Tx_n||^2 + (1-\alpha_n)||x_n - Tx_n||^2 \\ &\leq ||x_n - \bar{x}||^2 - (1-\alpha_n)[(1-\lambda) - (1-\alpha_n)]||x_n - Tx_n||^2 \\ &\leq ||x_n - \bar{x}||^2 \end{aligned} \tag{11}$$

So,

$$\begin{aligned} ||x_{n+1} - \bar{x}||^2 &= from\ (7) \\ &= ||(1-\alpha_n\mu_n)z_n + \alpha_n\mu_n(1-\alpha_n)(Tx_n - x_n) - \bar{x}||^2 \\ &= ||(1-\alpha_n\mu_n)(z_n - \bar{x}) + \alpha_n\mu_n[(1-\alpha_n)(Tx_n - x_n) - \bar{x}]||^2 \end{aligned}$$

from Lemma 2.5

$$\begin{aligned} &\leq (1-\alpha_n\mu_n)^2||z_n - \bar{x}||^2 + 2\alpha_n\mu_n\langle(1-\alpha_n)(Tx_n - x_n), j(x_{n+1} - \bar{x})\rangle \\ &\quad + 2\alpha_n\mu_n\langle \bar{-x}, j(x_{n+1} - \bar{x})\rangle \end{aligned}$$

from (11)

$$\begin{aligned} &\leq (1-\alpha_n\mu_n)||x_n - \bar{x}||^2 + 2\alpha_n\mu_n((1-\alpha_n)\langle Tx_n - x_n, j(x_{n+1} - \bar{x})\rangle \\ &\quad + \langle \bar{-x}, j(x_{n+1} - \bar{x})\rangle) \end{aligned} \tag{12}$$

Now , since x_n is bounded and $\omega_\ell(x_n) \subseteq Fix(T)$, there exists an appropriate subsequence $x_{n_k} \rightharpoonup p_0 \in Fix(T)$ such that

$$\limsup_n \langle -\bar{x}, j(x_{n+1} - \bar{x})\rangle = \lim_k \langle -\bar{x}, j(x_{n_k} - \bar{x})\rangle$$
$$= \langle -\bar{x}, j(p_0 - \bar{x})\rangle \leq 0 \tag{13}$$

From this, it follows that all the hypothesis of lemma 2.1 are satisfied and finally by (12). We conclude

$$\lim_{n\to\infty} ||x_n - \bar{x}|| = 0$$

Let now $\bar{x} \in Fix(T)$ be defined by the variational inequality (10).

Case II

If $||x_n - \bar{x}||$ does not be monotone nonincreasing, there exists a subsequence x_{n_k} such that

$$||x_{n_k} - \bar{x}|| < ||x_{n_{k+1}} - \bar{x}||, \forall k \in \mathbb{N}$$

So, by Lemma 2.2 , $\exists \tau(n) \uparrow +\infty$ such that

(1) $||x_{\tau(n)} - \bar{x}|| < ||x_{\tau(n)+1} - \bar{x}||$

(2) $||x_n - \bar{x}|| < ||x_{\tau(n)+1} - \bar{x}||$

Now, we have

$$\begin{aligned} 0 &\leq \liminf_n (||x_{\tau(n)+1} - \bar{x}|| - ||x_{\tau(n)} - \bar{x}||) \\ &\leq \limsup_n (||x_{\tau(n)+1} - \bar{x}|| - ||x_{\tau(n)} - \bar{x}||) \\ &\leq \limsup_n (||x_{n+1} - \bar{x}|| - ||x_n - \bar{x}||) \\ &\leq \limsup_n (||x_n - \bar{x}|| + \sqrt{\mu_n} M - ||x_n - \bar{x}||) = 0 \end{aligned}$$

Thus, we derive that

$$||x_{\tau(n)+1} - \bar{x}||^2 - ||x_{\tau(n)+1} - \bar{x}||^2 \to 0 \tag{14}$$

from which $||x_{\tau(n)} - Tx_{\tau(n)}|| \to 0$. Now, from (12).

We get

$$\begin{aligned} ||x_{\tau(n)+1} - \bar{x}||^2 &\leq (1 - \alpha_{\tau(n)}\mu_{\tau(n)}||x_{\tau(n)} - \bar{x}||^2 \\ &\quad + 2\alpha_{\tau(n)}\mu_{\tau(n)}(1 - \alpha_{\tau(n)})\langle Tx_{\tau(n)} - x_{\tau(n)}, j(x_{\tau(n)+1} - \bar{x})\rangle \\ &\quad + 2\alpha_{\tau(n)}\mu_{\tau(n)}\langle -\bar{x}, j(x_{\tau(n)+1} - \bar{x})\rangle \\ &= ||x_{\tau(n)} - \bar{x}||^2 + 2\alpha_{\tau(n)}\mu_{\tau(n)}(1 - \alpha_{\tau(n)})\langle Tx_{\tau(n)} \quad (15) \\ &\quad - x_{\tau(n)}, j(x_{\tau(n)+1} - \bar{x})\rangle \\ &\quad - 2\alpha_{\tau(n)}\mu_{\tau(n)}\langle -\bar{x}, j(x_{\tau(n)+1} - \bar{x})\rangle - 2\alpha_{\tau(n)}\mu_{\tau(n)}\frac{||x_{\tau(n)} - \bar{x}||^2}{2} \end{aligned} \tag{16}$$

putting in (15)

$$A_{\tau(n)} = (1-\alpha_{\tau(n)}\langle Tx_{\tau(n)}-x_{\tau(n)}, j(x_{\tau(n)+1}-\bar{x})\rangle-\langle-\bar{x}, j(x_{\tau(n)+1)}-\bar{x})\rangle-\frac{||x_{\tau(n)}-\bar{x}||^2}{2}$$

we have

$$||x_{\tau(n)+1}-\bar{x}||^2 \leq ||x_{\tau(n)}-\bar{x}||^2 + 2\alpha_{\tau(n)}\mu_{\tau(n)}A_{\tau(n)} \tag{17}$$

Notice that we cannot use Lemma 2.1 as in Case I (or in [12,13]). Since we cannot guarantee that $\sum_{n=1}^{\infty}\mu_{\tau(n)} = +\infty$. So, we proceed as follows Assume by contradiction that $||x_{\tau(n)}-\bar{x}||^2$ does not converge to 0. Then there exist (η_j) and an $\epsilon > 0$ such that

$$||x_{\tau(\eta_j)}-\bar{x}|| \geq 2\epsilon \tag{18}$$

By (11) and (12) we know that there exist $n_0, n_1 \in \mathbb{N}$ such that

$$(1-\alpha_{\tau(n)})\langle Tx_{\tau(n)}-x_{\tau(n)}, j(x_{\tau(n)+1}-\bar{x})\rangle < \frac{\epsilon}{3},\ \forall n \geq n_0 \tag{19}$$

and

$$\langle-\bar{x}, x_{\tau(n)+1}-\bar{x}\rangle < \frac{\epsilon}{3},\ \forall n \geq n_1 \tag{20}$$

Hence, if we take $\eta_{j_0} \geq \max n_0, n_1$ one obtains by definition of $A_{\tau(n)}$,

$$A_{\tau(n)} < \frac{\epsilon}{3}+\frac{\epsilon}{3}-\epsilon = \frac{-\epsilon}{3} < 0, \forall n \geq n_{j_0}$$

So, by (17) we have

$$||x_{\tau(n)+1}-\bar{x}||^2 \leq ||x_{\tau(n)}-\bar{x}||^2$$

which contradicts

$$||x_{\tau(n)}-\bar{x}|| \leq ||x_{\tau(n)+1}-\bar{x}||, \quad \text{for all } n$$

This implies that

$$||x_{\tau(n)}-\bar{x}|| \to 0$$

and so, using

$$||x_n-\bar{x}|| < ||x_{\tau(n)+1}-\bar{x}||$$

we finally obtain

$$||x_n-\bar{x}|| \to 0$$

In order to verify our main result we furnish the following example:

Example 3.2. The mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = -x$ is $\frac{1}{2}$-strictly pseudo-contractive. Taking $\alpha_n = \frac{1}{2}$, $\mu_n = \frac{1}{n}$, then our algorithm becomes

$$x_{n+1} = \frac{-1}{2n} x_n,$$

which gives $0 = Fix(T)$.

Remark 3.3. Our Theorem (3.1) improves and extends the main result of Marino et. al.[17] for a λ- strictly pseudocontractive mapping that solves a certain variational inequality in a more general uniformly smooth Banach space.

Remark 3.4. Our result extends the result of Yao et al. [27] from q-uniformly smooth Banach space to a more general uniformly smooth Banach space.

Remark 3.5. Our result improves the result of Shehu [23] in the sense that [23] proves the common fixed point of a finite family of strictly pseudocontractive mappings but our result includes the common fixed point of a λ- strictly pseudocontractive mapping and provides solution of a certain variational inequality in a uniformly smooth Banach space.

Open Questions

1. Is it possible to extend Theorem 3.1 for the family of λ-strictly pseudocontractive mappings?
2. Is it possible to design a more appropriate iteration scheme for proving strong convergence of λ-strictly pseudocontractive mapping in Theorem 3.1
3. Is it possible to prove Theorem 3.1 in a reflexive Banach Space having a uniformly Gateaux diferentiable norm?

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