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Cartesian Product of Two S-Valued Graphs

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Abstract

Motivated by the study of products in crisp graph theory and the notion of S-valued graphs, in this paper, we study the concept of cartesian product of two S-valued graphs.

Key words: Graph operations, Product of graphs, Semiring, S-valued graphs, vertex regularity, edge regularity.

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1. INTRODUCTION

Studies of how particular graphical parameters interact with graph products have lead several areas of research in graph theory. For example Shannon's capacity of a graph and Hedetniemi's coloring conjecture for the categorical product [6]. One of the oldest unsolved problems -Vizing's conjecture, also comes from the area of graph products. Algebraic graph theory can be viewed as an extension of graph theory in which algebraic methods are applied to problems about graphs [1]. Recently in [5] the authors have defined the concept of semiring valued graphs called S-valued graphs. In [3] they have studied the notion of regularity on S-valued graphs. In [2] the authors have studied the concept of vertex dominating set on S-valued graphs. Motivated by this, in this paper, we introduce the concept of cartesian product of two S-valued graphs and study some of its properties.

2. PRELIMINARIES

In this section, we recall the basic definitions that are needed for our work.

Definition 2.1. [6] The Cartesian product of G and H is a graph, denoted by $G\square H$ whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$. Thus

$$V(G\square H) = \{(g, h) | g \in V(G)\ and\ h \in V(H)\},$$

$$E(G\square H) = \{(g, h)(g', h') \big|\ g = g', hh' \in E(H) or gg' \in E(G), h = h'\}$$

Definition 2.2. [4] A semiring $(s, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations $+$ and $\cdot$ such that

(1) $(S, +, \cdot)$ is a monoid.

(2) $(S, \cdot)$ is a semigroup.

(3) For all $a, b, c \in S, a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b). c = a \cdot c + b \cdot c$

(4) $0 \cdot x = x \cdot 0 = 0$ for all $x \in S$.

The element 0 in S is called the additive identity as well as the zero of the semiring S.

Definition 2.3.[4] Let $(s, +, \cdot)$be a semiring. $\preceq$ is said to be a canonical pre-order if for a, b $\in$ S, a $\preceq$ b if and only if there exists an element c $\in$ S such that a + c = b.

Definition 2.4. [5] Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \emptyset$. For any semiring$(S, + \ \cdot)$, a semiring valued graph (or a S-valued graph) G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma: V \to S$ and $\psi : E \to S$ is defined by

$$\psi(x, y) = \begin{cases} \min\{\sigma(x), \sigma(y)\} & if\ \sigma(x) \preceq \sigma(y) \text{ or } \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

For every unordered pair (x, y) of $E \subset V \times V$.we call σ, a S-vertex set and ψ a S-edge set of the S-valued graph G^S.

Definition 2.5. [5] Let $G^S = (V, E, \sigma, \psi)$ be a S-valued graph. Then the graph $H^S = (P, L, \tau, \gamma)$ is called a S-subgraph of G^S if $P \subset V, L \subset E, \tau \subset \sigma$ and $\gamma \subset \psi$ .That is $\tau \subset \sigma \Rightarrow \tau(x) \preceq \sigma(x), x \in P$ and $\gamma \subset \psi \Rightarrow \gamma(x, y) \preceq \psi(x, y), (x, y) \in L \subset P \times P. H^S$ is called a S-sub graph of G^S is induced by P if $\tau(x) = \sigma(x)$ for every $x \in P$ and $\gamma(x, y) = \psi(x, y)$ for every $(x, y) \in L$.

Definition 2.6.[5] The open neighbourhood of v_i in G^S is defined as $N_S(v_i) = \{(v_j, \sigma(v_j)), where\ (v_i, v_j) \in E, \psi(v_i, v_j) \in S\}$ and the closed neighbourhood of v_i in $G^S$is defined as the set $N_S[v_i] = N_S(v_i) \cup \{(v_i, \sigma(v_i))\}$.

Definition 2.7. [5] The degree of a vertex v_i of the S-valued graph G^S is defined as

$deg_s(v_i) = \left(\sum_{v_j \in N_S(v_i)} \psi(v_i v_j), d(v_i)\right)$ where $d(v_i)$ is the number of edges incident with v_i.

Definition 2.8. [5] A S-valued graph G^S is said to be vertex degree regular S-valued graph (d_S - vertex regular graph) if $deg_S(v) = (a, n)$, for all $v \in V$ and some $a \in S$ and $n \in \mathbb{Z}^+$.

Definition 2.9. [5] $G^S = (V, E, \sigma, \psi)$ be a S-valued graph. If $\sigma(x) = a$, $\forall x \in V$ and for some $a \in S$ then the corresponding S-valued graph G^S is called a vertex regular S-valued graph (or simply vertex regular). G^S is said to be edge regular S-valued graph (simply edge regular) if $\psi(x, y) = a$ $\forall (x, y) \in E$ and for some $a \in S$. G^S is said to be a regular S-valued graph (S-regular) if it is both vertex regular and edge regular S-valued graph.

Definition 2.10. [5] A graph G^S is said to be (a, k) regular if the underlying crisp graph G is k-regular and $\sigma(v) = a$, $\forall\, v \in V$.

3. CARTESIAN PRODUCT OF TWO S-VALUED GRAPHS

In this section, we introduce the notion of Cartesian product of two S-valued graphs, illustrate with some examples, and prove simple properties.

Definition 3.1. Let $G_1{}^S = (V_1, E_1, \sigma_1, \psi_1)$ where $V_1 = \{v_i | 1 \le i \le p_1\}$, $E_1 \subset V_1 \times V_1$ and

$G_2{}^S = (V_2, E_2, \sigma_2, \psi_2)$ where $V_2 = \{v_2 | 1 \le j \le p_2\} E_2 \subset V_2 \times V_2$ be two given S-valued graphs.

$V_1 \times V_2 = \{w_{ij} = (v_i, u_j) | 1 \le i \le p_1, 1 \le j \le p_2\}$; $E_1 \times E_2 \subset V_1 \times V_2$.

The Cartesian product of two S-valued graphs $G_1{}^S$ and $G_2{}^S$ is a graph defined as

$G_\square{}^S = G_1{}^S \square G_2{}^S = (V = V_1 \times V_2, E = E_1 \times E_2, \sigma = \sigma_1 \times \sigma_2, \psi = \psi_1 \times \psi_2)$,

where $V = \{w_{ij} = (v_i, u_j) | v_i \in V_1$ and $u_j \in V_2$ and two vertices w_{ij} and w_{kl} are adjacent if $i = k$ and $u_j u_l \in E_2$ or $j = l$ and $v_i v_k \in E_1$.

Define $\sigma: V \to S$ by $\sigma(w_{ij}) = \min\{\sigma_1(v_i), \sigma_2(u_j)\}$ and $\psi: E \to S$ by

$$\psi(e_{ij}{}^{kl}) = \psi((v_i, u_j), (v_k, u_l))$$
$$= \begin{cases} \min\{\sigma_1(v_i), \psi_2(u_j, u_l)\} & if\ i = k\ and\ u_j u_l \in E_2 \\ \min\{\psi_1(v_i, v_k), \sigma_2(u_j)\} & if\ j = l\ and\ v_i v_k \in E_1 \end{cases}$$

Example 3.2. Consider the semiring $S = (\{0, a, b, c\}, +, \cdot)$with the binary operations '+' and '·' defined by the following Cayley tables.

+	0	a	b	c
0	0	a	b	c
a	a	b	c	c
b	b	c	c	c
c	c	c	c	c

•	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	c
c	0	c	c	c

In S we define a canonical pre-order $\preceq$ as follows:

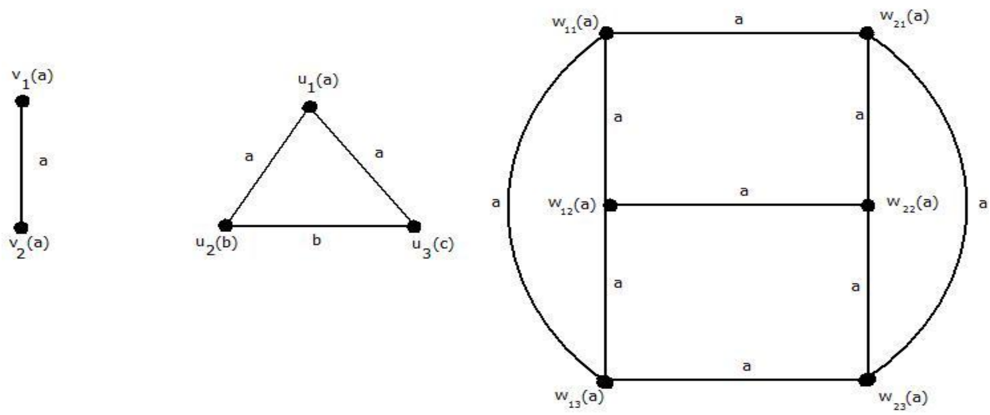
$0 \preceq 0, 0 \preceq a, 0 \preceq b, 0 \preceq c, a \preceq a, b \preceq b, c \preceq c, a \preceq b, a \preceq c, b \preceq c.$

Consider the two S-valued graphs ${G_1}^S$ and ${G_2}^S$

Then the Cartesian product ${G_\square}^S = {G_1}^S \square {G_2}^S$is given by ${G_1}^S \square {G_2}^S = (V, E, \sigma, \psi)$ where

$V = \{w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23},\}$,

$E = \{{e_{21}}^{11}, {e_{12}}^{11}, {e_{13}}^{11}, {e_{22}}^{12}, {e_{13}}^{12}, {e_{23}}^{13}, {e_{22}}^{21}, {e_{23}}^{21}, {e_{23}}^{22}\}$



Theorem 3.3. The Cartesian product of two S-regular graphs is S-regular.

Proof: Let ${G_1}^S = (V_1, E_1, \sigma_1, \psi_1)$ and ${G_2}^S = (V_2, E_2, \sigma_2, \psi_2)$ be two given S-regular graphs.

Claim:${G_\square}^S = {G_1}^S \square {G_2}^S$is S-regular.

Now by definition $\sigma(w_{ij}) = \min\{\sigma_1(v_i), \sigma_2(u_j)\} = \begin{cases} \sigma_1(v_i) \text{ } if \text{ } \sigma_1(v_i) \preceq \sigma_2(u_j) \\ \sigma_2(u_j) \text{ } if \text{ } \sigma_2(u_j) \preceq \sigma_1(v_i) \end{cases}$

Then in both the cases $\sigma(w_{ij})$ is equal for all $w_{ij} \in V, 1 \leq i \leq p_1, 1 \leq j \leq p_2$.

This implies that ${G_\square}^S = {G_1}^S \square {G_2}^S$is vertex S-regular.

Further, $\psi(e_{ij}{}^{kl}) = \begin{cases} \min\{\sigma_1(v_i), \psi_2(u_j, u_l)\} \; if \; i = k \; and \; u_j u_l \in E_2 \\ \min\{\psi_1(v_i, v_k), \sigma_2(u_j)\} \; if \; j = l \; and \; v_i v_k \in E_1 \end{cases}$

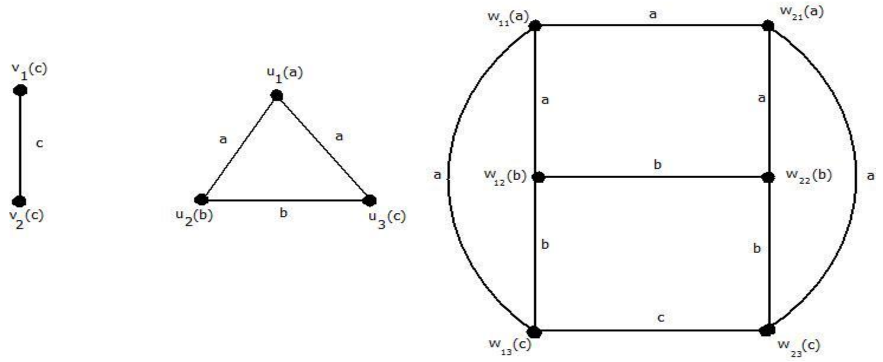
$$= \min\{\sigma_1(v_i), \sigma_2(u_j)\}$$

$$= \begin{cases} \sigma_1(v_i) \; if \; \sigma_1(v_i) \preceq \sigma_2(u_j) \\ \sigma_2(u_j) \; if \; \sigma_2(u_j) \preceq \sigma_1(v_i) \end{cases} \quad 1 \leq i, k \leq p_1, 1 \leq j, l \leq p_2$$

This implies that $G_{\square}{}^S = G_1{}^S \square G_2{}^S$is edge S-regular.

Thus the Cartesian product $G_1{}^S \square G_2{}^S$ is S-regular.

Remark 3.4. The converse of the above theorem need not be true in general as seen in example 3.2. Moreover consider the following example.



From the above example we observe that even if one of the S-valued graph is S-regular the product need not be S-regular. This leads to the following theorem.

Theorem 3.5. The product of two S-valued graphs is S-regular if the S-value corresponding to the S-regular graph is minimum among the S-values.

Proof: Let $G_{\square}{}^S = G_1{}^S \square G_2{}^S$ be S-regular.

Then $\sigma(w_{ij}) = \min\{\sigma_1(v_i), \sigma_2(u_j)\} = a, \; \forall i, j.$ (3.1)

Since$\sigma_1(v_i)$, $\sigma_2(u_j) \in S$, $\forall i, j$ either $\sigma_1(v_i) \preceq \sigma_2(u_j)$ or $\sigma_2(u_j) \preceq \sigma_1(v_i)$. By 3.1 if

$$\sigma_1(v_i) \preceq \sigma_2(u_j), \; \forall i, j \Rightarrow \sigma_1(v_i) = a \; , \; \forall i \quad (3.2)$$

and if $\sigma_2(u_j) \preceq \sigma_1(v_i) \; \forall i, j \Rightarrow \sigma_2(u_j) = a \; , \; \forall j$ (3.3)

Now consider any edge $e_{ij}{}^{kl}$. By definition $\psi(e_{ij}{}^{kl}) =$

$\begin{cases} \min\{\sigma_1(v_i), \psi_2(u_j, u_l)\} \; if \; i = k \; and \; u_j u_l \in E_2 \\ \min\{\psi_1(v_i, v_k), \sigma_2(u_j)\} \; if \; j = l \; and \; v_i v_k \in E_1 \end{cases}$

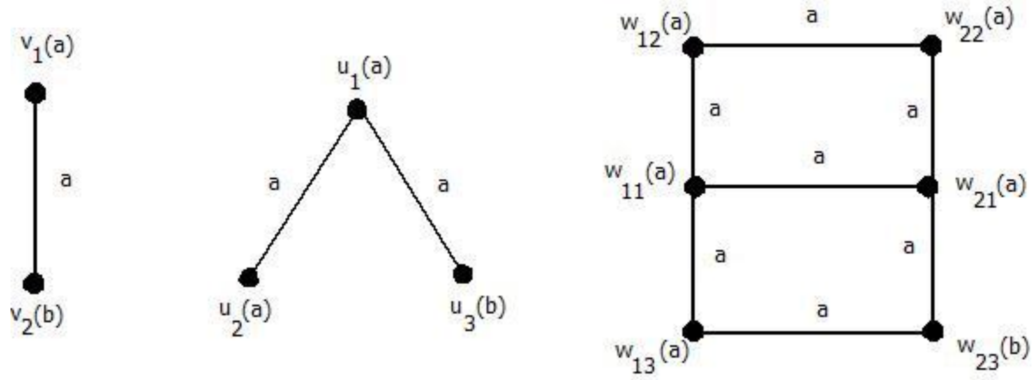
$$= \begin{cases} \min\{\sigma_1(v_i), \min\{\sigma_2(u_j), \sigma_2(u_l)\}\} \; if \; i = k \; and \; u_j u_l \; \in E_2 \\ \min\{\min\{\sigma_1(v_i), \sigma_1(v_k)\}, \sigma_2(u_j)\} \; if \; j = l \; and \; v_i v_k \; \in E_1 \end{cases}$$

$$= \min \begin{cases} \{\sigma_1(v_i), \sigma_2(u_j)\} \;\; if \sigma_2(u_j) \preceq \sigma_2(u_l) \\ \{\sigma_1(v_i), \sigma_2(u_l)\} \;\; if \sigma_2(u_l) \preceq \sigma_2(u_j) \\ \{\sigma_1(v_i), \sigma_2(u_j)\} \;\; if \sigma_1(v_i) \preceq \sigma_1(v_k) \\ \{\sigma_1(v_k), \sigma_2(u_j)\} \;\; if \sigma_1(v_k) \preceq \sigma_1(v_i) \end{cases} \qquad (3.4)$$

Among all the cases we obtain either $\sigma_1(v_i)$ or $\sigma_2(u_j)$ and both are equal to a, which is minimum among the S-values.

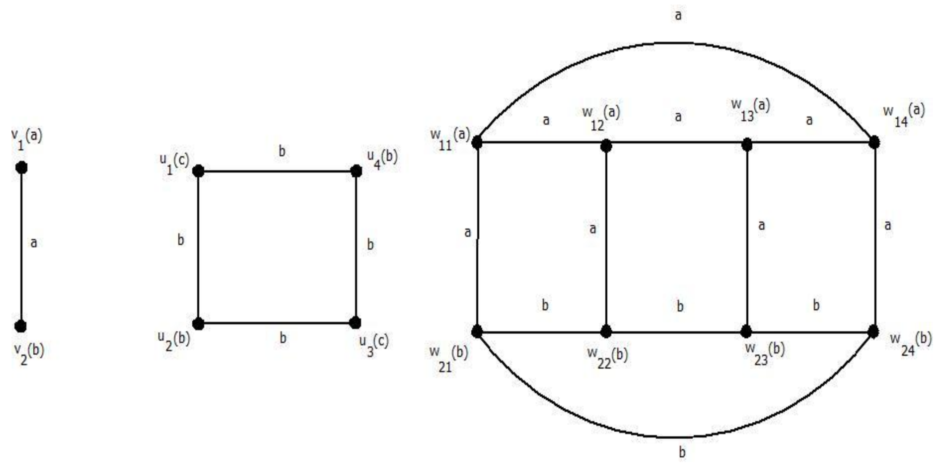
This proves that ${G_\square}^S$is S-regular whenever the S-value corresponding to the S-regular graph is minimum.

Remark 3.6. Product of two vertex S-regular graph is edge S-regular. But the converse is not true.



Clearly ${G_\square}^S$is edge S-regular but not vertex S-regular.

Example 3.7. The Cartesian product of two edge S-regular graphs is not edge S-regular. Consider the following two edge regular S-graphs.



It is clear that$G_1{}^S \square G_2{}^S$is not edge S-regular.

Theorem 3.8. The Cartesian product of two edge S-regular graphs is edge S-regular only if $\psi_1(e_i{}^k) = \psi_2(e_j{}^l)$ for $1 \leq i, k \leq p_1, 1 \leq j, l \leq p_2$.

Proof: Let $G_1{}^S$ be S-edge regular then$\psi_1(e_i{}^k) = \min\{\sigma_1(v_i), \sigma_1(v_k)\}$

$$= \begin{cases} \sigma_1(v_i) \; if \; \sigma_1(v_i) \preceq \sigma_1(v_k) \\ \sigma_1(v_k) \; if \; \sigma_1(v_k) \preceq \sigma_1(v_i) \end{cases} \quad (3.5)$$

Let $G_2{}^S$ be S-edge regular then$\psi_2(e_j{}^l) = \min\{\sigma_2(u_j), \sigma_2(u_l)\}$

$$= \begin{cases} \sigma_2(u_j) \; if \; \sigma_2(u_j) \preceq \sigma_2(u_l) \\ \sigma_2(u_l) \; if \; \sigma_2(u_l) \preceq \sigma_2(u_j) \end{cases} \quad (3.6)$$

Now $\psi(e_{ij}{}^{kl}) = \begin{cases} \min\{\sigma_1(v_i), \psi_2(u_{j,}u_l)\} \; if \; i = k \; and \; u_j u_l \in E_2 \\ \min\{\psi_1(v_i, v_k), \sigma_2(u_j)\} \; if \; j = l \; and \; v_i v_k \in E_1 \end{cases}$

$$= \begin{cases} \min\{\sigma_1(v_i), \min\{\sigma_2(u_j), \sigma_2(u_l)\}\} \; if \; i = k \; and \; u_j u_l \in E_2 \\ \min\{\min\{\sigma_1(v_i), \sigma_1(v_k)\}, \sigma_2(u_j)\} \; if \; j = l \; and \; v_i v_k \in E_1 \end{cases}$$

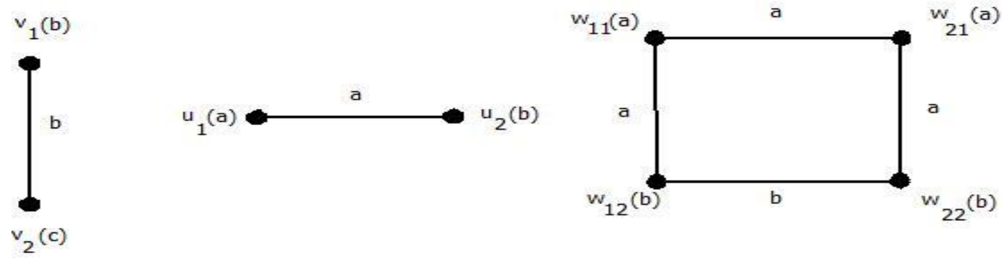
$$= \min \begin{cases} \{\sigma_1(v_i), \sigma_2(u_j)\} \; if \sigma_2(u_j) \preceq \sigma_2(u_l) \\ \{\sigma_1(v_i), \sigma_2(u_l)\} \; if \sigma_2(u_l) \preceq \sigma_2(u_j) \\ \{\sigma_1(v_i), \sigma_2(u_j)\} \; if \sigma_1(v_i) \preceq \sigma_1(v_k) \\ \{\sigma_1(v_k), \sigma_2(u_j)\} \; if \sigma_1(v_k) \preceq \sigma_1(v_i) \end{cases}$$

From 3.5 and 3.6 we observe that $G_\square{}^S$ isedge S-regular only when$\sigma_1(v_i) = \sigma_2(u_j)$ and $\sigma_1(v_k) = \sigma_2(u_l) \; \forall i, j, k, l$

Thus for ${G_\square}^S$ to be edge S-regular, we must have $\psi_1({e_i}^k) = \psi_2({e_j}^l)$ for $1 \leq i, k \leq p_1, 1 \leq j, l \leq p_2$.

Example 3.9. The Cartesian product of two d_S-regular graph is not d_S-regular.

Consider the following two d_S-regular graphs ${G_1}^S$ and ${G_2}^S$.



We observe that ${G_\square}^S$is not d_S-regular.

Theorem 3.10. If ${G_1}^S$ is $(a, m) -$ regular and ${G_2}^S$ is (b, n) $-$regular graph then their Cartesian product ${G_\square}^S$ is either $(a, m + n) -$ regular or $(b, m + n) -$ regular.

Proof: Let ${G_1}^S$ be $(a, m) -$ regular and ${G_2}^S$ be $(b, n) -$ regular for some $a, b \in S$ and $m, n \in \mathbb{Z}^+$

To prove:${G_\square}^S$ is $(a, m + n) -$ regular or $(b, m + n) -$ regular.

That is to prove ${G_\square}^S$ is vertex S-regular and $d(w_{ij}) = m + n$ for all $i.j$.

By theorem 3.3, ${G_\square}^S$ is vertex S-regular.

Further, the number of edges incident with w_{ij} in ${G_\square}^S$ is equal to the number of edges incident with v_i in ${G_1}^S$+ number of edges incident with u_j in ${G_2}^S$.

That is, $d(w_{ij}) = d(v_i) + d(u_j) = m + n = k$ (say) $\forall i, j$

Thus ${G_\square}^S$ is $(a, k) -$ regular.

4. CONCLUSION

Motivated by the study of S-valued graphs in [3] and [5], we studied the regularity and degree regularity conditions on the cartesian product of two S-valued graphs. In future, we have proposed to study the notions of minimal and maximal degree and their properties on ${G_\square}^S$.

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