On The Binding Number of Corona of Complete graph with Paths and their complements

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Abstract

The binding number of several interesting types of graphs has been studied extensively in the literature. The determination of the binding number of a graph $G$, in general seams to be a hard problem. So, in this paper we study the binding number of some special graphs. Precisely his paper determines the binding number of the corona of the complete graphs with the paths and their complements.

Keywords: Binding number of graphs, Corona of graphs, Complete graphs, Paths.

1. Introduction

Woodall 1973 defined the binding number of a graph as follows. If $S \subseteq V(G)$, we write the open neighbourhood of the set $S$ as $N(S) = \bigcup_{v \in S} N(v)$. The binding number of a graph $G$, denoted by $b(G)$, is given by

$$b(G) = \min_{S \in F} \frac{|N(S)|}{|S|},$$

where $F = \{S \subseteq V(G) : S \neq \emptyset, N(S) \neq V(G)\}$.

Woodall 1973 determined the binging number of the complete graphs, complete bipartite graphs, paths and cycles. (Kane et al.,1981) computed the binding number for a variety of graph products. (Kittrel et al., 1994) found the binding number of cubic $K$–hallian graphs.

The corona(Shakhatreh et al.,2006) of two graphs $G_1$ and $G_2$ denoted by $G_1 \circ G_2$ is defined to be the graph $G$ obtained by taking one copy of $G_1$ (which has $n_1$ vertices) and $n_1$ copies of $G_2$, and joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$. If $G_2$ has $n_2$ vertices , then it follows from the definition of the corona that $G_1 \circ G_2$ has $n_1(1+n_2)$ vertices and in general $G_1 \circ G_2$ is not isomorphic to $G_2 \circ G_1$. 
$K_n, K_{m,n}, P_n, C_n, cp(n)$ and $rK_2$ stand for complete graph, complete bipartite graph, path, cycle, cocktail party graph and Ladder graph. (AL-Tobaili, 2009) determined the binding number of $K_n o K_{m,n}$. (AL-Tobaili, 2008) determined the binding number of $K_n o K_n, rK_2 o K_n, cp(m) o K_n, P_n o K_n$ and $C_n o K_n$. Also, (AL-Tobaili, 2010) computed the binding number of $K_n o rK_2$ and $K_n o cp(n)$.

In this paper we continue the study of the binding numbers of the corona of graphs. Precisely we determine the binding number of the complete graphs with paths and their complements.

**Proposition 1** (Woodall 1973). If $G$ is a graph on $n$ vertices with minimum degree $\delta$, then

$$b(G) \leq \frac{n-1}{n-\delta}.$$ 

**2. Binding number of the corona of $K_n$ and $P_m$**

Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$, $n \geq 5$ and $V(P_m) = \{b_1, b_2, \ldots, b_m\}$, $m \geq 5$. Let $w \in V(K_n o P_m)$. Then $\deg(w) = m + n - 1$, if $w = v_i, 1 \leq i \leq n$ and

$$\deg(w) = \begin{cases} 2, & \text{if } w = b_i \text{ or } b_m, \\ 3, & \text{if } w = b_1, \ i \neq 1, m. \end{cases}$$

**Lemma 2.1** Let $m \geq n \geq 5$. Then $\forall X \subseteq V(K_n o P_m)$ such that $N(X) \neq V(K_n o P_m)$, $X \cap K_n \neq \Phi$ and $X \cap (\bigcup_{i=1}^m P_m) = \Phi$,

$$\min \frac{|N(X)|}{|X|} = \frac{mn + n - m}{n - 1}.$$ 

**Proof:** We have two cases:

(i) $|X \cap K_n| = 1$. Then $\frac{|N(X)|}{|X|} = m + n - 1$.

(ii) $|X \cap K_n| = l, \ 2 \leq l \leq n - 1$. $\frac{|N(X)|}{|X|} = \frac{n + lm}{l}$. 

Let \( l = n - 1 \). Then \( \frac{|N(X)|}{|X|} = \frac{n}{n-1} + m \).

Clearly \( \min \left\{ \frac{mn + n - m - n}{n-1}, m + n - 1 \right\} = \frac{mn + n - m}{n-1} \).

Lemma 2.2: Let \( m \geq n \geq 5 \). Then \( \forall X \subseteq V(K_n \circ P_m) \) such that \( N(X) \neq V(K_n \circ P_m) \), \( X \cap K_n = \emptyset \) and \( X \cap \bigcup_{i=1}^{r} P_{m_i} \neq \emptyset \),

\[
\min \left\{ \frac{|N(X)|}{|X|} \right\} = \begin{cases} 
\frac{2mn + 2n - m - 1}{2mn - m}, & \text{if } m \text{ is even,} \\
1, & \text{if } m \text{ is odd.} 
\end{cases}
\]

**Proof:** Since we are searching for the minimum of \( \frac{|N(X)|}{|X|} \) and since \( |N(X)| \) is increasing by increasing \( |X| \), so, we try to make \( |N(X)| \) and \( |X| \) as nearly as possible. This can be satisfied by the following cases:

**Case 1.** \( m \) is an even integer and \( X \) consists of all the vertices \( b_i : i \) is even from each of some \( l \) \((1 \leq l \leq n)\) copies of \( P_m \). So,

\[
|X| = \frac{lm}{2} \quad \text{and} \quad |N(X)| = \frac{lm}{2} + l. \quad \text{Therefore}
\]

\[
\frac{|N(X)|}{|X|} = \frac{m + 2}{m}.
\]

**Case 2.** \( m \) is an odd integer and \( X \) consists of all the vertices \( b_i : i \) is odd from each of some \( l \) \((1 \leq l \leq n)\) copies of \( P_m \). Then we get \( |X| = l(\frac{m+1}{2}) \) and

\[
|N(X)| = l(\frac{m-1}{2}) + l. \quad \text{Thus} \quad \frac{|N(X)|}{|X|} = 1.
\]

**Case 3.** Let \( n = l + k, l \geq 1, k \geq 1 \).

We consider two subcases:

(i) \( m \) is an even integer and without loss of generality let \( X \) consists of the whole vertices of the \( l \) copies of \( P_m \) and the vertices \( b_i : i \) is even from each of the \( k \)
copies of \( P_m \). So, \( |X| = ml + \frac{km}{2} \), \( |N(X)| = ml + \frac{km}{2} + k \). To, maximize \( X \), let

\( l = n-1 \) and \( k = 1 \). Thus
\[
\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m - 1}{2nm - m} < \frac{m + 2}{m}.
\]

For, if
\[
\frac{2nm + 2n - m - 1}{2nm - m} > \frac{m + 2}{m} \Rightarrow 1 + \frac{2n - 1}{m(2n - 1)} > \frac{2}{m} \Rightarrow 1 > 2, \text{ a contradiction.}
\]

(ii) \( m \) is an odd integer and without loss of generality let \( X \) consists of the whole vertices of the \( l \) copies of \( P_m \) and the vertices \( b_i : i \) is odd from each of the \( k \) copies of \( P_m \). So, \( |X| = ml + k \left( \frac{m+1}{2} \right) \), \( |N(X)| = ml + l + k \left( \frac{m-1}{2} \right) + k \). To, maximize \( X \), let \( l = n - 1 \) and \( k = 1 \). Thus

\[
\frac{|N(X)|}{|X|} = \frac{2nm - m + 2n - 1}{2nm - m + 1} = \frac{2mn - m + 1 + 2n - 2}{2mn - m + 1} = 1 + \frac{2(n-1)}{2mn - m + 1} > 1.
\]

This completes the proof of Lemma 2.2.

**Lemma 2.3:** Let \( m \geq n \geq 5 \). Then \( \forall X \subseteq V(K_n \circ P_m) \) such that \( N(X) \neq V(K_n \circ P_m) \), \( X \cap K_n \neq \Phi \) and \( X \cap \bigcup_{i=1}^{l} P_{m_i} \neq \Phi \),

\[
\min \frac{|N(X)|}{|X|} = \begin{cases} 
\frac{mn + n - m}{mn + n - m - 1}, & \text{if } m \text{ is even}, \\
1, & \text{if } m \text{ is odd}.
\end{cases}
\]

***Proof***: We consider two cases:

**Case(I):** \( X \cap K_n = \{ v_f \}, f \) is fixed, \( 1 \leq f \leq n \). We have the following sub-cases:

(a) For some fixed \( j, 1 \leq j \leq n \), \( X \cap P_{m_j} \neq \Phi \) and \( X \cap \bigcup_{i=1}^{l} P_{m_i} = \Phi, i \neq j \).

(a.1) \( f = j \). Then \( |X| = 1 + m, |N(X)| = n + m \). Thus

\[
\frac{|N(X)|}{|X|} = \frac{n + m}{m + 1}.
\]

(a.2) \( f \neq j \). Three sub-sub cases to be considered:

(a.2.1) \( P_{m_j} \subseteq X \). Then \( |X| = 1 + m, |N(X)| = n - 1 + m + m = 2m + n - 1 \). Therefore

\[
\frac{|N(X)|}{|X|} = \frac{n + 2m - 1}{m + 1}.
\]

(a.2.2) \( m \) is an even integer and \( X \) consists the even vertices from the copy \( P_{m_j} \).
On The Binding Number of Corona of Complete graph with Paths and their complements

Then $|X| = 1 + \frac{m}{2}$, $|N(X)| = n - 1 + \frac{m}{2}$. Thus

$$\frac{|N(X)|}{|X|} = \frac{2n + 3m - 2}{m + 2}.$$  

(a.2.3) $m$ is an odd integer and $X$ consists the odd vertices from the copy $P_m$.

Then $|X| = 1 + \frac{m+1}{2}$, $|N(X)| = n - 1 + \frac{m-1}{2}$. Therefore

$$\frac{|N(X)|}{|X|} = \frac{3m + 2n - 3}{m + 3}.$$  

(b) For some fixed $j$, $1 \leq j \leq n$, $X \cap P_{mj} = \Phi$ and $X \cap \left( \bigcup_{i=1}^{w} P_{mi} \right) \neq \Phi, i \neq j$.

(b.1) $f = j$.

(b.1.1) $X$ consists the whole vertices of some $l$ ($1 \leq l \leq n-1$) copies of $P_m$ of

$$\left( \bigcup_{i=1}^{w} P_{mi} \right) i \neq j.$$ Then $|X| = 1 + ml$, $|N(X)| = n - 1 + ml + m$. To maximize $X$, let $l = n - 1$. Hence

$$\frac{|N(X)|}{|X|} = \frac{nm + n - 1}{nm - m + 1}.$$  

(b.1.2) $m$ is an even integer and $X$ consists the even vertices of some 

$l$ ($2 \leq l \leq n-1$) copies of $P_m$ of

$$\left( \bigcup_{i=1}^{w} P_{mi} \right) i \neq j.$$ Then $|X| = 1 + \frac{lm}{2}$, $|N(X)| = n - 1 + \frac{lm}{2} + m$. To maximize $X$, let $l = n - 1$. So, 

$$\frac{|N(X)|}{|X|} = \frac{nm + 2n - 2}{nm - m + 2}.$$  

(b.1.3) $m$ is an odd integer and $X$ consists the vertices $b_i : i$ is odd from each of some $l$ ($2 \leq l \leq n-1$) copies of $P_m$ of

$$\left( \bigcup_{i=1}^{w} P_{mi} \right) i \neq j.$$ Then $|X| = 1 + l\left( \frac{m+1}{2} \right)$, $|N(X)| = n - 1 + l\left( \frac{m-1}{2} \right) + m$. To maximize $X$, let $l = n - 1$. Thus

$$\frac{|N(X)|}{|X|} = \frac{nm + n + m - 1}{nm + n - m + 1}.$$  

(b.2) $f \neq j$.

(b.2.1) $X \cap P_{mj} \neq \Phi$. 

To maximize $X$, let $l = n - 1$. Thus

$$\frac{|N(X)|}{|X|} = \frac{nm + n + m - 1}{nm + n - m + 1}.$$  

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(b.2.1) $X \cap P_{mj} \neq \Phi$. 

To maximize $X$, let $l = n - 1$. Thus

$$\frac{|N(X)|}{|X|} = \frac{nm + n + m - 1}{nm + n - m + 1}.$$  

(b.2) $f \neq j$.

(b.2.1) $X \cap P_{mj} \neq \Phi$.
(b.2.1.1) $X$ consists the whole vertices of some $l$ ($1 \leq l \leq n - 2$) copies of $P_m$ of $\left( \bigcup_{i=1}^{r} P_{ml} \right) i \neq j$. Then $|X|=1+m+ml$, $|N(X)|=n+m+ml$. To maximize $X$, let 
$$l = n - 2.$$ Therefore $$\frac{|N(X)|}{|X|} = \frac{nm+n-m}{nm-m+1}.$$ 

(b.2.1.2) $m$ is an even integer and $X$ consists the even vertices of some $l$ ($2 \leq l \leq n - 2$) copies of $P_m$ of $\left( \bigcup_{i=1}^{r} P_{ml} \right) i \neq j$. Then $|X|=1+\frac{lm}{2}+m$, $|N(X)|=n+\frac{lm}{2}+l+m$. To maximize $X$, let 
$$l = n - 2.$$ Thus $$\frac{|N(X)|}{|X|} = \frac{nm+2n}{nm+2}.$$ 

(b.2.1.3) $m$ is an odd integer and $X$ consists the odd vertices from each of some $l$ ($2 \leq l \leq n - 2$) copies of $P_m$ of $\left( \bigcup_{i=1}^{r} P_{ml} \right) i \neq j$. Then $|X|=1+l\left(\frac{m+1}{2}\right)+m$, $|N(X)|=n+l\left(\frac{m-1}{2}\right)+m$. To maximize $X$, let 
$$l = n - 2.$$ Hence $$\frac{|N(X)|}{|X|} = \frac{nm+n+2}{nm+n}.$$ 

(b.2.2) $X \bigcap P_{mf} = \Phi$. 

(b.2.2.1) $X$ consists the whole vertices of some $l$ ($1 \leq l \leq n - 2$) copies of $P_m$ of $\left( \bigcup_{i=1}^{r} P_{ml} \right) i \neq j$. Then $|X|=1+ml$, $|N(X)|=n-1+m+ml$. To maximize $X$, let 
$$l = n - 2.$$ So, $$\frac{|N(X)|}{|X|} = \frac{nm+n-m-1}{nm-2m+1}.$$ 

(b.2.2.2) $m$ is an even integer and $X$ consists the even vertices of some $l$ ($2 \leq l \leq n - 2$) copies of $P_m$ of $\left( \bigcup_{i=1}^{r} P_{ml} \right) i \neq j$. Then $|X|=1+\frac{lm}{2}$, $|N(X)|=n-1+\frac{lm}{2}+m$. To maximize $X$, let $l = n - 2$. Then 
$$\frac{|N(X)|}{|X|} = \frac{nm+2n-2}{nm-2m+2}.$$ 

(b.2.2.3) $m$ is an odd integer and $X$ consists the odd vertices from each of some $l$ ($2 \leq l \leq n - 2$) copies of $P_m$ of $\left( \bigcup_{i=1}^{r} P_{ml} \right) i \neq j$. Then
On The Binding Number of Corona of Complete graph with Paths and their complements

\[ |X| = 1 + l \left( \frac{m+1}{2} \right), |N(X)| = n - 1 + l \left( \frac{m-1}{2} \right) + m. \] To maximize \( X, \) let \( l = n - 2. \) Then \( \frac{|N(X)|}{|X|} = \frac{nm + n}{nm - n - 2m}. \)

(c) \( X \cap P_{mf} \neq \emptyset, \forall i = 1, 2, \ldots, m, i \neq f. \)

The union \( \bigcup_{i=1}^{r} P_{mi} \) contains \( n - 1 \) copies of \( P_{m}. \) Let us divided these \( n - 1 \) copies of \( P_{m} \) into two groups \( l \) and \( k. \) That is let \( n - 1 = l + k. \)

(c.1) \( X \cap P_{mf} \neq \emptyset. \)

(c.1.1) Let \( m \) be an even integer and let without loss of generality that \( X \) consists the whole vertices of the \( l \) copies of \( P_{m} \) and also consists the even vertices from each of the \( k \) copies of \( P_{m}. \) Then \( |X| = 1 + m + ml + \left( \frac{km}{2} \right), |N(X)| = n + m + ml + \left( \frac{km}{2} \right). \) Let \( l = n - 2 \) and \( k = 1. \) Then

\[ \frac{|N(X)|}{|X|} = \frac{2nm + 2n - m}{nm - m + 2}. \]

(c.1.2) Let \( m \) be an odd integer and let without loss of generality that \( X \) consists the whole vertices of the \( l \) copies of \( P_{m} \) and also consists the vertices \( b_i : i \) is odd from each of the \( k \) copies of \( P_{m}. \) Then

\[ |X| = 1 + m + ml + k \left( \frac{m+1}{2} \right), |N(X)| = n + m + ml + k \left( \frac{m-1}{2} \right). \] Let \( l = n - 2 \) and \( k = 1. \) Then

\[ \frac{|N(X)|}{|X|} = \frac{2nm + 2n - m - 1}{2nm - m + 3}. \]

(c.2) \( X \cap P_{mf} = \emptyset. \)

(c.2.1) \( X \) consists the whole vertices of the copies \( l \) and \( k. \) Then

\[ |X| = 1 + ml + mk, |N(X)| = n - 1 + m + ml + mk. \] So,

\[ \frac{|N(X)|}{|X|} = \frac{n - 1 + m(l + k)}{1 + m(l + k)}. \]

Let \( l + k = n - 1. \) Then

\[ \frac{|N(X)|}{|X|} = \frac{nm + n - 1}{nm - m + 1}. \]

(c.2.2) \( m \) is an even integer and the \( l \) copies contained in \( X \) and also \( X \) consists the even vertices from each of the \( k \) copies of \( P_{m}. \) Then
\(|X| = 1 + ml + \frac{km}{2}, \ |N(X)| = n - 1 + ml + \frac{km}{2}\). Let \(l = n - 2\) and \(k = 1\). Therefore

\[
\frac{|N(X)|}{|X|} = \frac{2nm + 2n - 3m - 2}{2nm - 3m + 2}.
\]

(c.2.3) \(m\) is an odd integer and \(X\) consists the whole vertices of the \(l\) copies of \(P_m\) and also consists the vertices \(b_j : i\) is odd from each of the \(k\) copies of \(P_m\). Then

\[|X| = 1 + ml + k\left(\frac{m + 1}{2}\right), \ |N(X)| = n - 1 + ml + k\left(\frac{m - 1}{2}\right)\]. Let \(l = n - 2\) and \(k = 1\). Thus

\[
\frac{|N(X)|}{|X|} = \frac{2nm + 2n - 3m - 3}{2nm - 3m + 3}.
\]

Case (II): \(X \cap K_n = \{v_1, v_2, \ldots, v_j\}, 2 \leq l \leq n - 1\). Then \(K_n \subseteq N(X)\). Divided the copies of \(P_m\) into two groups \(l\) and \(k\). That is \(n = l + k, l \geq 2, k \geq 1\).

(a) \(X\) does not contain any vertices from any copy of the \(k\) copies of \(P_m\). Then we take \(|X| = l + ml, \ |N(X)| = n + ml\). Let \(l = n - 1\) and \(k = 1\). Then

\[
\frac{|N(X)|}{|X|} = \frac{nm + n - m}{nm + n - m - 1}.
\]

(b) \(X\) contains vertices from each copy of some \(k\) copies of \(P_m\).

(b.1) \(m\) is an odd integer and \(X\) consists the vertices \(b_j : i\) is odd from each of some \(k\) copies of \(P_m\). Then \(|X| = l + ml + k\left(\frac{m + 1}{2}\right), \ |N(X)| = n + ml + k\left(\frac{m - 1}{2}\right)\). Let \(l = n - 1\) and \(k = 1\). Therefore

\[
\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m - 1}{2nm + 2n - 1}.
\]

(b.1.2) \(m\) is an even integer and \(X\) consists the even vertices from each of some \(k\) copies of \(P_m\). Then \(|X| = l + ml + \frac{km}{2}, \ |N(X)| = n + ml + \frac{km}{2}\). Let \(l = n - 1\) and \(k = 1\). Thus

\[
\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m}{2nm + 2n - m - 2}.
\]

(c) \(X\) consists the whole vertices of each copy of the \(k\) copies of \(P_m\) except one copy, say \(P_{mj}\) such that \(P_{mj} \not\subseteq X\).
(c.1) \( m \) is an even integer and \( X \) consists the even vertices from the copy \( P_{mj} \).

Then \( |X| = l + ml + (k-1)m + \frac{m}{2} \), \( |N(X)| = n + ml + (k-1)m + \frac{m}{2} \). Let \( l = n - 2 \) and \( k = 2 \). Then

\[
\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m}{2nm + 2n - 2m - 4}.
\]

(c.1.2) \( m \) is an odd integer and \( X \) consists the vertices \( b_i : i \) is odd from the copy \( P_{mj} \). Then \( |X| = l + ml + (k-1)m + \frac{m+1}{2} \), \( |N(X)| = n + ml + (k-1)m + \frac{m-1}{2} \). Let \( l = n - 2 \) and \( k = 2 \). Then

\[
\frac{|N(X)|}{|X|} = \frac{2nm + 2n - m - 1}{2nm + 2n - m - 3}.
\]

We collect all the results in case of \( m \) is an even together in the following claim.

**Claim 2.1:**

\[
\min \left\{ \frac{2n + 3m - 2}{m + 2}, \frac{mn + 2n + m - 2}{mn - m + 2}, \frac{mn + 2n}{mn + 2}, \frac{mn + 2n - 2}{mn - 2m + 2}, \frac{2mn + 2n - m}{2mn + 2n - m}, \frac{2mn + 2n - 3m - 2}{2mn - 3m + 2}, \frac{2mn + 2n - 3m - 2}{2mn - 3m + 2} \right\}
\]

\[
= \frac{2mn + 2n - 3m - 2}{2mn - 3m + 2}.
\]

**Proof:** We show that every element of the above set is greater than \( \frac{2mn + 2n - 3m - 2}{2mn - 3m + 2} \).

Suppose not, that is, every element of the above set is less than \( \frac{2mn + 2n - 3m - 2}{2mn - 3m + 2} \).

Let us take \( \frac{mn + 2n + m - 2}{mn - m + 2} < \frac{2mn + 2n - 3m - 2}{2mn - 3m + 2} \), implies

\[
4mn^2 + 4m^2n + 12m < 2mn^2 + 6m^2 + 8mn
\]

\[
\Rightarrow n^2 + 2mn + 6 < 3m + 4m, \text{ a contradiction, since}
\]

\( m \geq n \geq 5 \).

In a similar manner we can show for the rest elements of the above set.

Similarly we collect all the results in case of \( m \) is an odd together in the following claim.
Claim 2.2:
\[
\min \left\{ 1, \frac{2mn + 2n - m - 1}{2mn - m + 3}, \frac{mn + n + 1}{mn + n - 2m}, \frac{mn + n}{m + 3}, \frac{3m + 2n - 3}{2mn + 2n - m - 1} \right\} = 1.
\]

**Proof:** Proof is similar to the proof of Claim 2.1.

Also, collect all the results in case of there is no restriction on \(m\) together in the following claim.

Claim 2.3:
\[
\min \left\{ \frac{m + n}{m + 1}, \frac{2m + n - 1}{m + 1}, \frac{mn - m}{m + 1}, \frac{mn + n - m - 1}{mn - m - 1}, \frac{mn + n - m - 1}{mn - m + 1}, \frac{mn + n - m - 1}{mn + n - m - 1} \right\} = \frac{mn + n - m - 1}{mn + n - m - 1}.
\]

**Proof:** Proof is similar to the proof of Claim 2.1.

From Claim 2.1 and Claim 2.3, we have

Claim 2.4:
\[
\min \left\{ \frac{2mn + 2n - 3m - 2}{2mn - 3m + 2}, \frac{mn + n - m}{mn + n - m - 1} \right\} = \frac{mn + n - m}{mn + n - m - 1}.
\]

**Proof:** Proof is similar to the proof of Claim 2.1.

Also, by Claim 2.2 and Claim 2.3, we have

Claim 2.5:
\[
\min \left\{ 1, \frac{mn + n - m}{mn + n - m - 1} \right\} = 1.
\]

This completes the proof of Lemma 2.3.

By Lemma 2.1, Lemma 2.2 and Claim 2.4, we have

Claim 2.6:
\[
\min \left\{ \frac{mn + n - m}{n - 1}, \frac{2mn + 2n - m - 1}{2mn - m}, \frac{mn + n - m}{mn + n - m - 1} \right\} = \frac{mn + n - m}{mn + n - m - 1}.
\]

**Proof:** Proof is similar to the proof of Claim 2.1.

Also, by Lemma 2.1, Lemma 2.2 and Claim 2.5, we have
On The Binding Number of Corona of Complete graph with Paths and their complements

Claim 2.7:
\[
\min \left\{ \frac{mn + n - m}{n - 1}, \frac{2mn + 2n - m - 1}{2mn + 2n - m}, \frac{2mn + 2n - m - 1}{2mn + 2n - 1} \right\} = \frac{2mn + 2n - m - 1}{2mn + 2n - 1}.
\]

Proof: Proof is similar to the proof of Claim 2.1.

Theorem 2.1 Let \( m \geq n \geq 5 \). Then
\[
b(K_n \circ P_m) = \begin{cases} 
\frac{mn + n - m}{mn + n - m - 1}, & \text{if } m \text{ is even}, \\
1, & \text{if } m \text{ is odd}.
\end{cases}
\]

Proof: By Lemma 2.1, Lemma 2.2, Lemma 2.3, Claim 2.6 and Claim 2.7, theorem follows.

3. Binding number of the corona of \( K_n \) and \( \overline{P_m} \)

Clearly two vertices \( b_i, b_j \in V(\overline{P_m}) \) (\( 1 \leq i, j \leq m \)) are nonadjacent if and only if \( |i - j| = 1 \) and adjacent if and only if \( |i - j| \geq 2 \). Also,

\[
|N(b_i, b_j)| = \begin{cases} 
m - 2, & \text{if } |i - j| = 1, \\
m - 1, & \text{if } |i - j| = 2, \\
m, & \text{if } |i - j| \geq 3.
\end{cases}
\]

and \( |N(b_i, b_{i+1}, b_{i+2})| = m - 1(1 \leq i \leq m) \).

Lemma 3.1 Let \( m \geq n \geq 7 \). Then
\[
\forall X \subseteq V(K_n \circ \overline{P_m}) \text{ such that } N(X) \neq V(K_n \circ \overline{P_m}), X \cap K_n \neq \Phi \text{ and }
\]
\[
X \cap \left( \bigcup_{i=1}^{n} \overline{P_m} \right) = \Phi, \ \min \frac{|N(X)|}{|X|} = \frac{mn + n - m}{n - 1}.
\]

Proof: Proof is similar to the proof of Lemma 2.1.

Lemma 3.2 Let \( m \geq n \geq 7 \). Then \( \forall X \subseteq V(K_n \circ \overline{P_m}) \) such that \( N(X) \neq V(K_n \circ \overline{P_m}), X \cap K_n = \Phi \) and \( X \cap \left( \bigcup_{i=1}^{n} \overline{P_m} \right) \neq \Phi \), \( \min \frac{|N(X)|}{|X|} = \frac{mn + n - 1}{mn - n + 3} \).
Proof: We consider the following cases:

(a) $X$ consists only three vertices of the form $b_i, b_{i+1}, b_{i+2}$ from each of some $l (1 \leq l \leq n)$ copies of $\overline{P_m}$. Then $|X| = 3l$, $|N(X)| = l(m - 1) + l$. Thus
\[
\frac{|N(X)|}{|X|} = \frac{m}{3}.
\]

(b) $X$ consists at least two vertices $b_i, b_j$ such that $|i - j| \geq 3$ from each of some $l (1 \leq l \leq n)$ copies of $\overline{P_m}$. Then $|X| = ml$, $|N(X)| = ml + l$. Thus
\[
\frac{|N(X)|}{|X|} = \frac{m + 1}{m}.
\]

(c) $X$ consists at least two vertices $b_i, b_j$ such that $|i - j| \geq 3$ from each some $l$ copies of $\overline{P_m}$ and $X$ consists only three vertices of the form $b_i, b_{i+1}, b_{i+2}$ $(1 \leq i \leq m)$ from each of some $k (2 \leq l + k \leq n, l, k \geq 1)$ copies of $\overline{P_m}$. Then $|X| = ml + 3k$, $|N(X)| = ml + l + k(m - 1) + k$. Let $l = n - 1$ and $k = 1$. Thus
\[
\frac{|N(X)|}{|X|} = \frac{mn + n - 1}{mn - n + 3}.
\]

It can be easy seen that
\[
\min \left\{ \frac{m}{3}, \frac{m + 1}{m}, \frac{mn + n - 1}{mn - n + 3} \right\} = \frac{mn + n - 1}{mn - n + 3}.
\]

Lemma 3.3 Let $m \geq n \geq 7$. Then $\forall X \subseteq V(K_n \circ \overline{P_m})$ such that $N(X) \neq V(K_n \circ \overline{P_m})$, $X \cap K_n \neq \Phi$ and $X \cap \left( \bigcup_{i=1}^{n} \overline{P_m} \right) \neq \Phi$, \( \min \frac{|N(X)|}{|X|} = \frac{mn + n - m}{mn + n - m - 1} \).

Proof: Consider two cases:

Case(I): $X \cap K_n = \{v_f\}$, for some fixed $f, 1 \leq f \leq n$.

(I.1) \( X \cap \overline{P_{mj}} \neq \Phi \), for some fixed $j, 1 \leq j \leq m$ and $X \cap \left( \bigcup_{i=1}^{n} \overline{P_{mi}} \right) = \Phi$, $i \neq j$.

(I.1.1) $f = j$. Then $|X| = m + 1$, $|N(X)| = m + n$. Thus
\[
\frac{|N(X)|}{|X|} = \frac{m + n}{m + 1} > \frac{mn + n - m}{mn + n - m - 1} \quad \ldots \ldots \mathrm{(I)}.
\]
For, if
\[
\frac{m+n}{m+1} < \frac{mn+n-n}{mn+n-m-1},
\]
\[
\Rightarrow mn^2 + n^2 - mn - n < mn + n
\]
\[
\Rightarrow mn - m + (n - 1) < m + 1
\]
\[
\Rightarrow mn < 2m + 1
\]
\[
\Rightarrow n < 2 + \frac{1}{m}
\]
\[
\Rightarrow n < 3, \text{ a contradiction.}
\]

(I.1.2) \( f \neq j \).

(I.1.2.1) \( X \) consists only three consecutive vertices from the copy \( \overline{P_{mf}} \).

Then \( |X| = 1 + 3, |N(X)| = n - 1 + m - 1 + m \). Thus
\[
\frac{|N(X)|}{|X|} = \frac{2m + n - 2}{4} > \frac{mn + n - m}{mn + n - m - 1} \quad \text{......(2)}.
\]

(I.1.2.2) \( X \) consists at least two vertices \( b_x, b_y \) such that \( |x - y| \geq 3 \) from the copy \( \overline{P_{mf}} \). Then \( |X| = m + 1, |N(X)| = n - 1 + m + m \). Thus
\[
\frac{|N(X)|}{|X|} = \frac{2m + n - 1}{m + 1} > \frac{mn + n - m}{mn + n - m - 1} \quad \text{......(3)}.
\]

(I.2) \( X \cap \left( \bigcup_{i=1}^{w} \overline{P_{mf}} \right) \neq \Phi \).

(I.2.1) \( \overline{P_{mf}} \cap X \neq \Phi \).

(I.2.1.1) \( X \) consists only three consecutive vertices from each of some \( l(1 \leq l \leq n-1) \) copies other than the copy \( \overline{P_{mf}} \). Then \( |X| = 1 + m + 3l, |N(X)| = n + m + l(m - 1) \). Let \( l = n - 1 \). Thus
\[
\frac{|N(X)|}{|X|} = \frac{mn + 1}{3n + m - 2} > \frac{mn + n - m}{mn + n - m - 1} \quad \text{......(4)}.
\]

(I.2.1.2) \( X \) consists at least two vertices \( b_x, b_y \) such that \( |x - y| \geq 3 \) from each of some \( l(1 \leq l \leq n-2) \) copies other than the copy \( \overline{P_{mf}} \). Then \( |X| = 1 + m + lm, |N(X)| = n + m + lm \). Let \( l = n - 2 \). For if, \( l = n - 1 \), implies that
\[ N(X) = V(K_n \circ \overline{P}_m). \] Thus

\[ \frac{|N(X)|}{|X|} = \frac{mn + n - m}{mn - m + 1} > \frac{mn + n - m}{mn + n - m - 1} \] ......(5).

\[ \text{(I.2.1.3)} \] \( X \) consists at least two vertices \( b_x, b_y \) such that \(|x - y| \geq 3\) from each of some \( l \) copies of \( \overline{P}_m \) and \( X \) consists only three consecutive vertices from each of some \( k \) copies of \( \overline{P}_m \) \((2 \leq l + k \leq n - 1, l, k \geq 1)\). Then

\[ |X| = 1 + lm + 3k, |N(X)| = n + lm + k(m - 1). \] Let \( l = n - 2 \) and \( k = 1 \). Thus

\[ \frac{|N(X)|}{|X|} = \frac{mn + n - m - 1}{mn - m + 4} > \frac{mn + n - m}{mn + n - m - 1} \] ......(6).

\[ \text{(I.3)} \] \( \overline{P}_{mf} \cap X = \emptyset. \)

\[ \text{(I.3.1)} \] \( X \) consists at least two vertices \( b_x, b_y \) such that \(|x - y| \geq 3\) from each of some \( l \) \((1 \leq l \leq n - 1)\) copies of \( \overline{P}_m \). \( |X| = 1 + lm, |N(X)| = n - 1 + lm. \) Let \( l = n - 1 \). Thus

\[ \frac{|N(X)|}{|X|} = \frac{mn + n - m - 1}{mn - m + 1} > \frac{mn + n - m}{mn + n - m - 1} \] ......(7).

\[ \text{(I.3.2)} \] \( X \) consists only three consecutive vertices from each of some \( l \) copies of \( \overline{P}_m \) \((1 \leq l \leq n - 1)\). \( |X| = 1 + 3l, |N(X)| = n - 1 + l(m - 1). \) Let \( l = n - 1 \). Thus

\[ \frac{|N(X)|}{|X|} = \frac{mn - m}{3n - 2} > \frac{mn + n - m}{mn + n - m - 1} \] ......(8).

\[ \text{(I.3.3)} \] \( X \) consists at least two vertices \( b_x, b_y \) such that \(|x - y| \geq 3\) from each of some \( l \) copies of \( \overline{P}_m \) and \( X \) consists only three consecutive vertices from each of some \( k \) copies of \( \overline{P}_m \) \((2 \leq l + k \leq n - 1, l, k \geq 1)\). Then

\[ |X| = 1 + ml + 3k, |N(X)| = n - 1 + ml + k(m - 1). \] Let \( l = n - 2 \) and \( k = 1 \). Thus

\[ \frac{|N(X)|}{|X|} = \frac{mn + n - m - 2}{mn - 2m + 4} > \frac{mn + n - m}{mn + n - m - 1} \] ......(9).

**Case(II):** \( X \cap K_n = \{v_1, v_2, \ldots, v_l\}, 2 \leq l \leq n - 1. \) Then \( K_n \subseteq N(X). \) Let there exist at least one vertex, say, \( v_f \in V(K_n) \) and \( v_f \notin K_n \cap X, \) otherwise \( N(X) = V(K_n \circ \overline{P}_m). \)

Clearly, the vertices \( v_1, v_2, \ldots, v_l \) are joined to the copies \( \overline{P}_{m1}, \overline{P}_{m2}, \ldots, \overline{P}_{ml} \) respectively.

Let there are \( k \) copies of \( \overline{P}_m \) that are different from the \( \overline{P}_{m1}, \overline{P}_{m2}, \ldots, \overline{P}_{ml} \).
\(2 \leq l + k \leq n - 1\) copies. Consider the copy \(\overline{P_{mf}}\).

\textbf{(II.1)} \(X\) consists at least two vertices \(b_x, b_y\) such that \(|x - y| \geq 3\) from each of the \(k\) copies of \(\overline{P_m}\).

\textbf{(II.1.1)} \(X \cap \overline{P_{mf}} = \Phi\). Then \(|X| = l + ml + mk, |N(X)| = n + ml + mk\).

\textbf{(II.1.1.1)} \(k = 0\). Thus

\[
\frac{|N(X)|}{|X|} = \frac{n + ml}{l + ml},
\]

To maximize \(X\), let \(l = n - 1\). So,

\[
\frac{|N(X)|}{|X|} = \frac{mn + n - m}{mn - n - m - 1} \quad \ldots\ldots\text{(10)}.
\]

\textbf{(II.1.1.2)} \(k \geq 1\). Then \(k = n - l - 1\). Therefore

\[
\frac{|N(X)|}{|X|} = \frac{n + mn - m}{mn - m + l}.
\]

To maximize \(X\), let \(l = n - 2\). Hence

\[
\frac{|N(X)|}{|X|} = \frac{n + mn - m}{mn - m + n - 2} > \frac{mn + n - m}{mn + n - m - 1} \quad \ldots\ldots\text{(11)}.
\]

\textbf{(II.1.2)} \(X \cap \overline{P_{mf}} = \{b_i, b_{i+1}, b_{i+2}\}\). Then

\(|X| = l + ml + mk + 3, |N(X)| = n + ml + mk + m - 1\).

\textbf{(II.1.2.1)} \(k = 0\). Take \(l = n - 1\). Then

\[
\frac{|N(X)|}{|X|} = \frac{mn + n - 1}{mn + n - m + 2} > \frac{mn + n - m}{mn + n - m - 1} \quad \ldots\ldots\text{(12)}.
\]

\textbf{(II.1.2.2)} \(k \geq 1\). Then as in (II.1.1.2), we get

\[
\frac{|N(X)|}{|X|} = \frac{mn + n - 1}{mn + n - m + 1} > \frac{mn + n - m}{mn + n - m - 1} \quad \ldots\ldots\text{(13)}.
\]

\textbf{(II.2)} \(X\) consists only three consecutive vertices from each of the \(k\) copies of \(\overline{P_m}\).

\textbf{(II.2.1)} \(X \cap \overline{P_{mf}} = \Phi\). Then \(|X| = l + ml + 3k, |N(X)| = n + ml + k(m - 1)\). Let \(l = n - 2\) and \(k = 1\). Thus

\[
\frac{|N(X)|}{|X|} = \frac{mn + n - m - 1}{mn - n - 2m + 1} > \frac{mn + n - m}{mn + n - m - 1} \quad \ldots\ldots\text{(14)}.
\]

\(X \cap \overline{P_{mf}} = \{b_i, b_{i+1}, b_{i+2}\}\). Then
\[|X| = l + ml + 3k + 3, \quad |N(X)| = n + ml + k(m-1) + m - 1. \]  Let \( l = n - 2 \) and \( k = 1 \).

Thus
\[
\frac{|N(X)|}{|X|} = \frac{mn + n - 2}{mn + n - 2m + 4} > \frac{mn + n - m}{mn + n - m - 1}
\] ......(15).

**Case (II.3)** Let \( k = a + b, a \geq 1, b \geq 1 \). Let \( X \) consists at least two vertices \( b_x, b_y \) such that \(|x - y| \geq 3\) from each of the \( a \) copies and only three consecutive vertices form each of the \( b \) copies of \( P_m \). Thus
\[|X| = l + ml + ma + 3b, \quad |N(X)| = n + ml + ma + b(m - 1). \] To maximize \( X \), let \( a = b = 1 \). Then
\[
\frac{|N(X)|}{|X|} = \frac{mn + n - m - 1}{mn + n - 2m} > \frac{mn + n - m}{mn + n - m - 1}
\] ......(16).

The proofs of (2) up to (16) is similar to (1).

Thus by (1) up to (16), Lemma 3.3 follows.

From Lemma 3.1, Lemma 3.2 and Lemma 3.3, we have the following claim.

**Claim 3.1** \( \min \left\{ \frac{mn + n - m}{n - 1}, \frac{mn + n - 1}{mn - n + 3}, \frac{mn + n - m}{mn + n - m - 1} \right\} = \frac{mn + n - m}{mn + n - m - 1} \).

**Proof:** Assume on the contrary that
\[
\frac{mn + n - m}{n - 1} < \frac{mn + n - m}{mn + n - m - 1}
\]
\[\Rightarrow m^2 n^2 - 2m^2 n + mn^2 - mn + m^2 < 0 \]
\[\Rightarrow m^2 n^2 - 2m^2 n + mn^2 - mn < 0 \]
\[\Rightarrow m^2 n^2 + mn^2 < 2m^2 n + mn \]
\[\Rightarrow mn + n < 2m + 1 \]
\[\Rightarrow mn < 2m + 1 \]
\[\Rightarrow n < 2 + \frac{1}{m} \]
\[\Rightarrow n < 3, \]

A contradiction.

Similarly we can show that \( \frac{mn + n - 1}{mn - n + 3} > \frac{mn + n - m}{mn + n - m - 1} \).

Hence the claim is proved.

**Theorem 3.1** Let \( m \geq n \geq 7 \). Then
\[
b(K_o \circ P_m) = \frac{mn + n - 1}{mn + n - m - 1}.
\]
**Proof:** For small values of $m$ and $n$ the theorem can easy to prove. Let $m \geq n \geq 7$. Then by Lemma 3.1, Lemma 3.2, Lemma 3.3, and Claim 3.1, theorem follows.

**References**


