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On Co-Multiplication Gamma Acts

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Abstract

The purpose of this paper is to introduce the notion of co-multiplication gamma acts which is the dual notion of multiplication gamma acts. Some results and characterization related this notion obtained. Various properties concerning multiplication and co-multiplication gamma acts over a commutative gamma semigroups have been discussed and the relationship between them has been studied. The concept of second gamma subact was investigated and study the relation between it and co-multiplication gamma acts.

Keywords: Co-multiplication Gamma Acts, Gamma Acts, Gamma semigroups, Multiplication Gamma Acts, Second Gamma Subacts.

1. Introduction

In 1988, Sen MK, Saha [1] introduced the concept of gamma semigroups as a generalization of semigroups. They studied the structure of gamma semigroups and obtained various generalizations analogous of corresponding parts in semigroup theory. Later a lot of researchers studied on the gamma semigroups. Chinram R, Jirojkul C [2] extended many classical notions of semigroups to Γ-semigroups and they studied the properties of Γ-semigroups. In 2016, Abbas M.S, Faris [3] introduced the notion of gamma acts over gamma semigroup as generalization of acts over semigroups. Many classical concepts and results of the theory of acts have been extended and generalized to gamma acts. They also study several properties such as: ordered gamma acts, quotient gamma acts, free gamma acts and cofree gamma acts of gamma acts supported by examples and counter examples. In 2021, Abbas MS, Jubeir [4] introduced the concept of multiplication gamma acts. Many properties and results about these concept was studied and another concepts have been studied such as: α-divisible gamma acts, large gamma acts, uniform gamma acts, gamma nilpotent, fully invariant gamma subacts and finitely cogenerated gamma acts and their properties are clarified. Also, used the concept of multiplication gamma acts to introduce a new mathematical system which is called a semigroup associative with multiplication gamma act and its basic properties are discussed. Moreover, they showed that under the condition of multiplication gamma acts many properties of gamma acts can be done. As a continuation of these studies, definition of pure gamma act and idempotent gamma acts are given in [5, 6].

In our work the notion of co-multiplication gamma acts was introduced as a dual notion of multiplication gamma acts and investigated some main properties of this class of gamma acts. Some results concerning multiplication and co-multiplication gamma acts over a commutative gamma semigroups obtained. Furthermore, another characterization for co-multiplication gamma acts was given. Some examples and counterexamples have been considered. Also characterize certain class of gamma semigroups (acts) such as co-multiplication gamma acts and second gamma subacts, in terms of co-multiplication gamma acts. Thought this paper S will be denote a gamma semigroup with zero element.

2. Basic Concepts

In this section concepts relating to this work are defined.

Definition 2.1. [1]: Let S, Γ be non-empty sets, S is called a gamma semigroup (briefly, Γ-semigroup) if there exists a mapping: $S \times \Gamma \times S \to S$ defined (s, α, t) by $s\alpha t$ that satisfies the condition: $s\alpha(t\beta r)=(s\alpha t)\beta r$ for all $s, t, r \in$ S and $\alpha, \beta \in \Gamma$. A Γ-semigroup S is said to be commutative if $s_1\alpha s_2=s_2\alpha s_1$ for all $s_1, s_2 \in$ S and $\alpha \in \Gamma$.

Definition 2.2.[1]: Let S be a Γ-semigroup. An element $s_1 \in$ S is called the left (right) identity of S if $s_1\alpha s_2 = t$ ($s_2\alpha s_1 = t$) for all $s_2 \in S$ and $\alpha \in \Gamma$. An element $s \in$ S is called identity if it is both a left and right identity of S. A Γ-semigroup with identity is called a Γ-monoid. The identity of a Γ-semigroup (if exists) is denoted by 1.

Definition 2.3. [1]: Let S be a Γ-semigroup. A non-empty subset T of S is said to be a Γ-subsemigroup of S if $t_1\alpha t_2 \in$ T for all $t_1, t_2 \in$ T and $\alpha \in \Gamma$.

Definition 2.4 .[1]: Let S be a Γ-semigroup. A non-empty subset K of Γ-semigroup S is called left (right) Γ-ideal if SΓK ⊆ K (KΓS ⊆K) where SΓK=$\{s\alpha k\colon s \in S, \alpha \in \Gamma$ and $k \in K\}$. The term "Γ-ideal" refers to a two-sided Γ-ideal.

Proposition 2.5 [1]: The union of any family of Γ-ideals of Γ-semigroup S is a Γ-ideal of S.

Proposition 2.6.[7]: Let S be a Γ-semigroup. A Γ-ideal Q of S is called prime if for every two Γ-ideals K, L of S with KΓL ⊆ Q, either K ⊆ Q or L ⊆ Q.

Proposition 2.7.[7]: Let S be a Γ-semigroup. A Γ-ideal K of S is called maximal if it is proper and is not properly contained in any proper Γ-ideal of S.

Definition 2.8.[8]: Let S be Γ-semigroup and α ∈ Γ. An element a in S is said to be left (right) α-cancellative if $a\alpha b = a\alpha c$ ($b\alpha a = c\alpha a$) implies $b = c$ for all $b, c \in$ S.

Definition 2.9.[8]: Let S be Γ-semigroup. An element a in S is said to be left (right) Γ-cancellative if a is left (right) α-cancellative for all α ∈ Γ.

Definition 2.9.[3]: Let S be a Γ-semigroup. A non-empty set A is called left gamma act over S (briefly, S_Γ-act) if there exists a mapping: $S \times \Gamma \times A \rightarrow A$ defined by $(s, \alpha, a) \mapsto s\alpha a$, satisfying $(s_1 \alpha s_2)\beta a = s_1 \alpha (s_2 \beta a)$ for all $s_1, s_2 \in$ S, $\alpha, \beta \in \Gamma$ and $a \in A$.Similarly, right gamma acts can be defined.
The concept of acts over semigroups is generalized to the following:

Definition 2.10.[3]: A nonempty subset B of left S_Γ-act A is said to be gamma subact (briefly, S_Γ-subact) if for all $s \in$ S, $\alpha \in \Gamma$ and $b \in B$ implies that $s\alpha b \in B$.

Definition 2.11.[3]: Let A be a S_Γ-act. An element $\theta \in A$ is said to be a zero of A if $s\alpha\theta = \theta$ and if S is a Γ-semigroup with zero then $0\alpha a = \theta$ for all $a \in$ A and $\alpha \in \Gamma$.

Proposition 2.12.[3]: Let $\{B_i\}_{i \in I}$ be a family of S_Γ-subacts of S_Γ-act A. Then, $\bigcup_{i \in I} B_i$ is S_Γ-subact of A, and if $\bigcap_{i \in I} B_i$ is nonempty then $\bigcap_{i \in I} B_i$ is S_Γ-subact of A.
For S_Γ-acts A and B. Amapping
f:A → B is called S_Γ-homomorphism if $f(s\alpha a) = s\alpha f(a)$ for all $s \in S$, $\alpha \in \Gamma$ and $a \in A$. If f is surjective, then f is S_Γ-epimorphism.

Definition 2.13.[3]: Let ρ be a equivalence relation on S_Γ-act A. Then ρ is called a congruence if for all $(m_1, m_2) \in \rho$ then $(s\alpha m_1, s\alpha m_2) \in \rho$ for all $s \in S$, $\alpha \in \Gamma$.
The quotient gamma act of the congruence ρ on A is denoted by A/ρ define by A/ρ= $\{a\rho \mid a \in$ A and $a\rho$ the equivalent class containing$a\}$.

Definition 2.14.[9]: Let S be Γ-monoid and A be a S_Γ-act. For $s \in$ S and $\alpha \in \Gamma$. An element $a \in$A is called α-divisible by s in A if there exists $b \in$ A such that $s\alpha b = a$.

An S_Γ-act A is said to be α-divisible if $c\alpha$A = A for any right α-cancellative element $c \in$ S. More generally, S_Γ-act A is called a Γ-divisible if $c\alpha$A = A for all right Γ-cancellative element $c \in$ S.

Definition 2.15.[3]: A proper S_Γ-subact B of S_Γ-act A is a prime if for any a ∈ A and s ∈ S, the set inclusion sΓSΓa ⊆ B implies either a ∈ B or s ∈ [B:$_S$ A].

Definition 2.16.[9]: Let A be a S_Γ-act and C be a S_Γ-subact of A . A prime S_Γ-subact B of A is called a minimal prime of C if C ⊆ B and there is no prime S_Γ-subact B′ such that C ⊆ B′ ⊂ B. A prime Γ-ideal P of Γ-semigroup S is minimal if it's minimal prime S_Γ-subact of S_Γ-act S.

Proposition 2.17.[8]: Let A be an S_Γ-act, and ρ bs a congruence on A. Then

- **i.** if B ≤ A, then B / ρ_B ≤ A / ρ
- **ii.** if W ≤ A / ρ, then there exist L ≤ A such that W= L / ρ_L, where ρ_L=ρ⋂(L × L).

Definition 2.18.[10]: Let S be a Γ-monoid and A be a S_Γ-act. Then the set all of its S_Γ-endomorphisms $End_S(A)$ is a Γ-monoid with respect to the product given by mapping: $End_S(A)$ ×Γ× $End_S(A)$ ⟶ $End_S(A)$ defined by $(f, \alpha, g) \mapsto f\ \alpha\ g$, where $(f\alpha g)$(a)=$g(1\alpha f$(a)) where $f, g \in End_S(A)$, $\alpha \in \Gamma$ and $a \in A$.

Definition 2.19.[10]: Let S be a Γ-semigroup and A be a S_Γ-act with $s,\ r \in$ S. Then A is called

- **i.** Faithful if the equality $s\alpha a = r\alpha a$ implies that $s = r$ for every a ∈A and α ∈ Γ.
- **ii.** Strongly faithful (briefly, st-faithful) if for $\alpha \in \Gamma$ the equality $s\alpha a = r\alpha a$ for some $a \in$ A implies that $s = r$.
- **iii.** Globally faithful (briefly, gl-faithful) if the equality $s\Gamma a = r\Gamma a$ for all $a \in$ A implies that $s = r$.

Let B and C be S_Γ-subacts of S_Γ-act A. Then the residual Γ-ideal of B by C is defined as, $[B:_S C]=\{s \in S \mid s\alpha c \in B$ for all $\alpha \in \Gamma$ and $c \in$ C}. In a special case in which B=θ, the Γ-ideal [θ:$_S$ C] is called annihilator of C and it is denoted by ann_S(C). For the Γ-ideal K of S the S_Γ-subact [θ:$_A$K] is called the annihilator of K in A and it is denoted by ann_A(K).

Definition 2.20.[4]: Let A be a S_Γ-act over Γ-semigroup S. Then A is called multiplication S_Γ-act if for all S_Γ-subact B of A has the form B=KΓA for some Γ-ideal K of S. Equivalently, S_Γ-act A is multiplication if and only if B= [B:$_S$ A]ΓA for all S_Γ-subact of A.
Consider A is a S_Γ-act and Q is a maximal Γ-ideal of Γ-monoid S, then we define: $T_Q(A)=\{a \in A\colon a = q\alpha a$ for some $q \in$ Q and $\alpha \in \Gamma\}$. Clearly, T_Q(A) is an S_Γ-subact of A. An S_Γ-act A is Q-cyclic provided there exist $p \in$ S\Q such that $p\Gamma A \subseteq S\Gamma a$, for all $a \in$ A. [4]

Theorem 2.21.[4]: Let S be a Γ-monoid. Then A is a multiplication S_Γ-act if and only if for every maximal Γ-ideal Q of S, either A= T_Q(A) or A is Q-cyclic.

Proposition 2.22.[4]: Let S be a Γ-monoid and A be a faithful S_Γ-act. Then A is a multiplication if and only if KΓA is a multiplication S_Γ-subact of A for all multiplication Γ-ideal K of S
The following Lemma that will needed in our work.

Lemma 2.23.[8]: Let K be a Γ-ideal of Γ-semigroup S. Then for any collection of Γ-ideals { L_i: i∈ I} of S, we have

i. $[K : \bigcup_{i\in I} L_i] = \bigcap_{i\in I}[K : L_i]$.

ii. $[\bigcap_{i\in I} L_i : K] = \bigcap_{i\in I}[L_i : K]$.

3. Main Results.

In this section the notion of a multiplication gamma acts was dualize to obtain a co-multiplication gamma acts and basic related properties which are introduced in our work

Definition 3.1. An S_Γ-act A is said to be a co-multiplication if for all S_Γ-subact B of A there exists a Γ-ideal K of Γ- semigroup S, such that B = $[\theta :_A K]$.

Examples 3.2.

i. Let S=A= $\{x, y, z\}$and Γ=$\{\alpha, \beta\}$. Then A is an S_Γ-act by the mapping: S ×Γ× A → A defined by the following tables:

α	x	y	z
x	x	x	x
y	x	x	x
z	x	y	x

β	x	y	z
x	x	x	x
y	x	x	y
z	x	y	x

Here, $B_1=\{x\}$, $B_2=\{x, y\}$ and B_3=A are S_Γ-subacts of A. It can be easily seen that A is co-multiplication.

ii. Let S =Γ=A={ i, 0, -i}. Then S is a Γ-semigroup under the multiplication over complex numbers. Here B_1= {0} and B_2= S, are the only Γ-ideals of S. Thus, A is co-multiplication.

iii. Let S = A ={ ∅, {a}, {b}, {a ,b}} and Γ={∅ , {a}}. Then A is an S_Γ-act by the mapping: S × Γ × A → A written by (A, B, C) ⟼ A⋂B⋂C. Here, A is not co-multiplication, since N{∅ ,{b}} is S_Γ-subact of A and all Γ-ideals of S are: {∅},{∅,{a}}, S with N ≠ [∅:$_A$ K] for every Γ-ideals K of S.

4. For all a congruence ρ on S_Γ-act A. Then A is a co-multiplication if and only if A/ρ is S_Γ-act.

Lemma 3.3. An S_Γ-act A is a co-multiplication if and only if for each S_Γ-subact B of A, B = $[\theta :_A ann_S(B)]$.

Proof:

(⟹) Clear.

(⟸) Suppose that A is a co-multiplication. Then there exists a Γ-ideal K of S, such that B=$[\theta :_A K]$. Thus, K⊆ $ann_S(B)$ so that $[\theta :_A ann_S(B)] \subseteq [\theta :_A K]$= B. This yields that B = $[\theta :_A ann_S(B)]$.

Example 3.4. Let S= $\Gamma = \mathbb{Z}$ and A= $\mathbb{Z}_8$. Then A is S$_\Gamma$-act under the usual of multiplication of integer numbers. Let B = $\{\bar{0}, \bar{4}\}$ be S$_\Gamma$-subact of A. Clearly, [$\bar{0}$: ann$_S$(B)]= S . Thus, S is not a co-multiplication.
The following example shows that not every co-multiplication S$_\Gamma$-act is a multiplication S$_\Gamma$-act.

Example 3.5. Let S= Γ= $\mathbb{Z}$. Then S is an S$_\Gamma$-act. Consider B=SΓ(2) as S$_\Gamma$-subact of S$_\Gamma$-act S. Then B=[0:$_S$ ann$_S$(B)]. Therefore, S is not a co-multiplication S$_\Gamma$-act. But $\mathbb{Z}$ is a multiplication S$_\Gamma$-act.

Proposition 3.6. Let A be an S$_\Gamma$-act and B_1, B_2 be co-multiplication S$_\Gamma$-subacts of A then $B_1 \cap B_2$ is co-multiplication
S$_\Gamma$-subact of A.

Proof: Let B_1=[θ:$_A$ ann$_S$(B_1)] and B_2= [θ:$_A$ ann$_S$(B_2)]. Then $B_1 \cap B_2$= [θ:$_A$ ann$_S$(B_2)] ∩ [θ:$_A$ ann$_S$(B_2)] = [θ:$_A$ ann$_S$(B_1)∪ ann$_S$(B_2)]. Hence, $B_1 \cap B_2$ is a co-multiplication S$_\Gamma$-subact of A.

Proposition 3.7. Let T be a Γ-subsemigroup of a Γ-semigroup S and A be an S$_\Gamma$-act. If A is a co-multiplication T_Γ-act, then A is a co-multiplication S$_\Gamma$-act.
Proof: Let B be S$_\Gamma$-subact of A. Then B is a T$_\Gamma$-subact of A. Thus B= [θ :$_A$ K] for some Γ-ideal K of T. Hence, A is a co-multiplication S$_\Gamma$-act.

Lemma 3.8. Let A, B be a S$_\Gamma$-acts and ϕ:A ⟶ B be a S$_\Gamma$-epimorphism with K is a Γ-ideal of a Γ-semigroup S. Then ϕ([θ :$_A$ K])=[θ :$_B$ K].

Proposition 3.9. Every homomorphic image of a co-multiplication S$_\Gamma$-act is a co-multiplication.
Proof: Let A, B be a S$_\Gamma$-acts and ϕ: A ⟶ B be an S$_\Gamma$-epimorphism. Assume that, A is co-multiplication and C is an S$_\Gamma$-subact of B. Then ϕ^{-1}(C) is S$_\Gamma$-subact of A, then there exists a Γ-ideal K of S such that ϕ^{-1}(C) = [θ :$_A$ K]. Hence, $\phi(\phi^{-1}$(C)) = ϕ([θ :$_A$ K]). Thus, by Lemma 3.8; K= [θ :$_B$ K].

Theorem 3.10. Let S be a Γ-monoid and A be a S$_\Gamma$-act. Then the following are equivalent.
Let A be a S$_\Gamma$-act and f : A ⟶ A be a S$_\Gamma$-endomorphism on A then we say that f is a S$_\Gamma$-homothety if there exists s∈ S such that $f(a) = s\alpha a$ for every a ∈ A and $\alpha \in \Gamma$.
Let B be a S$_\Gamma$-subact of S$_\Gamma$-act A. Then B is called second S$_\Gamma$-subact of A if for each s ∈ S the S$_\Gamma$-homothety f_s:B⟶B is either surjective or zero.

- **i.** A is a co-multiplication S$_\Gamma$-act.
- **ii.** For all S$_\Gamma$-subact B of A and Γ-ideal K of a Γ-semigroup S with B⊂ [θ :$_A$ K], there exists a Γ-ideal L of S such that K ⊂ L and B=[θ:$_A$ L].
- **iii.** For every S$_\Gamma$-subact B of A and Γ-ideal K of a Γ-semigroup S with B ⊂ [θ :$_A$ K], there exists a Γ-ideal L of S such that K⊂L and B⊆ [θ :$_A$ L].

Proof:

(i) $\Longrightarrow$ (ii) Let B be a S_Γ-subact of A and K be a Γ-ideal of a Γ-semigroup S, such that B⊂$[\theta:_A$ K]. Since A is a co-multiplication, B= [$\theta:_A$ ann_S(B)]. Suppose L=K ⋃ ann_S(B). Since B= [$\theta:_A$ ann_S(B)] ⊂ [$\theta:_A$ K], ann_S(B) ⊄ K. Hence, K ⊂ L and we have [θ $:_A$ L] =[$\theta:_A$ K ⋃ ann_S(B)] = [θ $:_A$ K] ⋂ [θ $:_A$ ann_S(B)]=B.

(ii) $\Longrightarrow$ (iii) Clear.

(iii) $\Longrightarrow$ (i) Let B be a S_Γ-subact of A and H={D: D is a Γ-ideal of S and B⊂ [$\theta:_A$ D]}. Clearly, 0 ∈ H. Let {K_i : i ∈ I } be any non-empty collection of Γ-ideals in H. Thus, $\bigcup_{i\in I} K_i$ ∈ H. By the Zorn's Lemma, H has a maximal member P such that B ⊆ [0 $:_A$ P]. Assume that B ≠[0 $:_A$ P]. Then by (iii), there exists a Γ-ideal L with P ⊂ L and B ⊆ [0 $:_A$ L]. But this is a contradiction by the choice of P. Thus, B =[0$:_A$P] and hence A is a co-multiplication S_Γ-act.

Theorem 3.11. Let S be a commutative Γ-monoid and A be a co-multiplication S_Γ-act. If B is a S_Γ-subact of A such that ann_S(B) is a prime Γ-ideal of S, then B is a second S_Γ-subact of A.

Proof: Since A is a co-multiplication S_Γ-act, then B= [$\theta:_M$ ann_S(B)]. Let s ∈ S and f_s: B→B be the nonzero S_Γ-homomorphism defined by $f_s(a) = s\alpha a$ for every a ∈ A and $\alpha \in \Gamma$. Let D = Im f_s = sΓB. Clearly, D ⊆ B. By Theorem 3.10, there exists a Γ-ideal K of S, such that ann_S(B) ⊂ K and D = [θ $:_A$ K]. Thus $K\Gamma(s$ΓB) =KΓD= θ. It follows that KΓs⊆ ann_S(B). Since ann_S(B) is a prime Γ-ideal of S and ann_S(B) ⊂ K, we have s ∈ ann_S(B) so that sΓB = θ; a contradiction. Hence, B is a second S_Γ-subact of M.

Corollary 3.12. Let S be a commutative Γ-semigroup and A be a co-multiplication S_Γ-act. Then B is second S_Γ-subact of A if and only if ann_S(B) is a prime Γ-ideal of S.

Proposition 3.13. Let A be a co-multiplication S_Γ-act.

i. Let $\{B_i, i\in I\}$ be a nonempty collection of S_Γ-subacts of S_Γ act A with $\bigcap_{i\in I} B_i$ = θ. Then for every S_Γ-subact B of A, we have B =$\bigcap_{i\in I}(B \cup B_i)$.

ii. Let P be a minimal Γ-ideal of S, such that [θ $:_A$ P] = θ. Then A is cyclic.

Proof:

i. Let B be a S_Γ-subact of A. Then B= [θ $:_A$ ann_S(B)]=[$\bigcap_{i\in I} B_i$ $:_A$ ann_S(B)]= $\bigcap_{i\in I}[B_i :_A ann_S$(B)]. Now, let $x\in B \cup B_i$, then $x \in B$ *or* $x\in B_i$. If $x \in B$ then ann_S(B)Γx⊆ B_i. Thus $x\in [B_i{:_A}\, ann_S$(B)]. If $x\in B_i$, then ann_S(B)Γ$x \subseteq B_i$ and hence $x\in[B_i{:_A}\, ann_S$(B)]. Thus, $\bigcap_{i\in I}(B_i \cup B) \subseteq \bigcap_{i\in I}[B_i :_A ann_S$(B)]=B. Also, B ⊆ $\bigcap_{i\in I}(B_i \cup B)$.Therefore, B=$\bigcap_{i\in I}(B_i \cup B)$.

ii Let $\theta \neq a$∈ A. Since A is a co-multiplication S_Γ-act, there exists a Γ-ideal K of S such that SΓa =[$\theta:_A K$]. Thus SΓa = [$\theta:_A K$] = [[$\theta:_A P$]$:_A$ K] = [θ $:_A$ PΓK]. Since P is a minimal Γ-ideal of S and θ ⊆ PΓA ⊆ P, we have PΓA = θ or PΓA = P. If PΓA = P, then SΓa=[$\theta:_A$ PΓA]=[$\theta:_A$ P] = θ. Hence $a = \theta$; a contradiction. If PΓA = θ, then SΓa=A. Therefore, A is cyclic.

Theorem 3.14. Let S be a Γ-semigroup and A be a co-multiplication S_Γ-act. Then the following are hold.

- **i.** Every S_Γ-subact of A is co-multiplication.
- **ii.** Every S_Γ-subact of A is fully invariant.
- **iii.** If A is a gl-faithful S_Γ-act, then A is Γ-divisible.

Proof:

- **i.** Let A be a co-multiplication S_Γ-act and B be a S_Γ-subact of A. If C is a S_Γ-subact of B, then there exists a Γ-ideal K of Γ-semigroup S such that K= $[\theta:_A K]$= $[\theta:_B K]$. Therefore, B is co-multiplication S_Γ-subact.
- **ii.** Let B be a S_Γ-subact of a co-multiplication S_Γ-act A. Then there exists a Γ-ideal K of S such that B=$[\theta:_A K]$. Suppose that f: A→ A be an S_Γ-endomorphism. Since KΓB = θ, K ⊆ $ann_S(f(B))$. Then $[\theta:_A ann_S(f(B))] \subseteq [\theta:_A K]$= A. This implies that $f(B) \subseteq$ B.
- **iii.** Let $c \in$ S be a right Γ-cancellative element. Then $c\Gamma A$=$[\theta:_A K]$ for some Γ-ideal K of S. Thus $K\Gamma(c\Gamma A) = \theta = 0\Gamma A$. By hypothesis, $K\Gamma c = 0$. Since c is right Γ-cancellative then K=0. Hence, $c\Gamma A = A$.

Proposition 3.15. Let S be a commutative Γ-monoid and A be a S_Γ-act. Then the set all S_Γ-endomorphisms of A, $End_S(A)$ is a commutative Γ-monoid.

Proof:

Let $f, g \in End_S(A)$ and let $a \in$ A. Then $f(a) \in f(S\Gamma a)$ and $g(a) \in g(S\Gamma a)$. By part (i), $f(S\Gamma a) \subseteq S\Gamma a$ and $g(S\Gamma a) \subseteq S\Gamma a$. So, $f(a), g(a) \in S\Gamma a$. Thus there exist $s, t \in$S and $\alpha, \beta \in\Gamma$ such that $f(a) = s\alpha a$ and $g(a) = t\beta a$. Hence,

$(fg)(a) = f(g(a)) = f(t\alpha a) = t\alpha f(a) =$
$t\alpha(s\gamma a) = (s\alpha t)\gamma a = s\alpha(t\gamma a) = s\alpha g(a) =$
$g(s\alpha a) = g(f(a)) = (gf)(a)$. Therefore, $End_S(A)$ is commutative Γ-monoid.

Proposition 3.16. Let S be a Γ-monoid and A be S_Γ-act

- **i.** If A is a co-multiplication S_Γ-act and B is a S_Γ-subact of A with $ann_S(B)$ = $ann_S(A)$, then A is a simple S_Γ-act.
- **ii.** If A is a multiplication S_Γ-act A and B is a S_Γ-subact of with [B:A]=S. Then A is a simple S_Γ-act.

Proof:

- **i.** By hypothesis and Lemma 3.3, B = $[\theta:_A ann_S(B)]=[\theta:_A ann_S(A)]$ = A. Therefore, A is a simple S_Γ-act.
- **ii.** Since A is a multiplication S_Γ-act then B=[B:A]ΓA. Thus B=[B:A]ΓA=SΓA=A.

Corollary 3.17. Let S be a Γ-monoid and A be a faithful multiplication and co-multiplication S_Γ-act. Then A is simple.

Proposition 3.18. Let S be a Γ-semigroup and A be a co-multiplication S_Γ-act with a nonempty collection of S_Γ-subacts $\{B_i, i \in I\}$ of A. Then $[\theta:_A \bigcap_{i\in I} ann_S(B_i)]$ $=\bigcup_{i\in I}[\theta:_A ann_S(B_i)]$.

Proof: By hypothesis, $B_i[\theta:_A ann_S(B_i)]$ for all $i\in I$. Also $\bigcup_{i\in I} B_i=[\theta:_A ann_S(\bigcup_{i\in I} B_i)]$. Now, by Lemma 2.23; $\bigcup_{i\in I} B_i=[\theta:_A ann_S(\bigcup_{i\in I} B_i)]=[\theta:_A \bigcap_{i\in I} ann_S(B_i)]$. On other hand $\bigcup_{i\in I} B_i=\bigcup_{i\in I}[\theta:_A ann_S(B_i)]$. So, $[\theta:_A \bigcap_{i\in I} ann_S(B_i)]=\bigcup_{i\in I}[\theta:_A ann_S(B_i)]$.

Corollary 3.19. Let A be a co-multiplication S_Γ-act and P,Q be a maximal Γ-ideals of a Γ-semigroup S with $[\theta:_A P]$, $[\theta:_A Q]$ are nonzero S_Γ-subacts of A. Then $[\theta:_A P]\cup[\theta:_A Q]=[\theta:_A P\cap Q]$.

Proof: Put K=$[\theta:_A P]$. Then PΓK=θ and hence P$\subseteq ann_S$(K). Since P is maximal Γ-ideal of S, then P =ann_S(K). Similarly, if L=$[\theta:_A Q]$, Q= ann_S(K). By Proposition 3.18, $[\theta:_A P]\cup[\theta:_A Q]=[\theta:_A P\cap Q]$.

Lemma 3.20. Let S be a Γ-monoid and B be a S_Γ-subact of S_Γ-act A. Then$\left[\theta:_A[0:_S[B:_S A]]\right]$ = [B:$_S$ A]ΓA.

Proof: $\left[\theta:_A[0:_S[B:_S A]]\right]\subseteq A=S\Gamma A$ =(SΓ[B:$_S$ A])ΓA $\subseteq$ [B:$_S$ A]ΓA. Let $x\in$[B:$_S$ A]ΓA, then $x=s\alpha a$ such that
$s\in$ [B:$_S$ A], $\alpha\in\Gamma$ and $a\in$ A. Now, $[0:_S[B:_S A]]\Gamma x=[0:_S[B:_S A]]\,\Gamma\,(s\alpha m)$ $=\left([0:_S[B:_S A]]\Gamma s\right)\alpha m=0\alpha m=\theta$. Thus, $x\in\left[\theta:_A[0:_S[B:_S A]]\right]$.

Theorem 3.21. Let S be a Γ-monoid and A be a multiplication S_Γ-act. Then A is co-multiplication S_Γ-act. The convers is true if S is co-multiplication as an S_Γ-act.

Proof:Let B be S_Γ-subact of A. Then B= [B :$_S$A]ΓA. By Lemma 3.20, $[\theta:_A[0:_S[B:_S A]]]$ =[B :$_S$ A]ΓA=B. Thus A is co-multiplication.

Conversely, since S is co-multiplication thenann_S(B)=$ann_S\left(ann_S\left(ann_S(B)\right)\right)$. Since A is a co-multiplication S_Γ-act and By Lemma 3.20, B=[θ :$_A$ ann_S(B)]=[θ :$_A$ $ann_S\left(ann_S\left(ann_S(B)\right)\right)$]=$\left(ann_S\left(ann_S(B)\right)\right)$ΓA. Thus, A is a multiplication S_Γ-act.

Corollary 3.22. If A is a multiplication S_Γ-act, then KΓA is a co-multiplication S_Γ-act for all multiplication Γ-ideal K of Γ-monoid S.

Proof: As is easily by Proposition 2.22 and Theorem 3.21.

Theorem 3.23. Let S be a Γ-monoid and A be an S_Γ-act. Consider the following conditions:

i. A is a co-multiplication S_Γ-act
ii. B=[θ :$_A$ ann_S(B)] for each S_Γ-subact B of A.
iii. For S_Γ-subact K and L of A; ann_S(K) ⊆ ann_S(L), then L⊆K.
iv. For S_Γ-subact K and $a\in$A; ann_S(K) ⊆ ann_S(SΓa), then $a\in K$.
v. For S_Γ-subact B of A and $a\in$ A with ann_S(B) ⊆ ann_S(SΓa), then [B :$_S$ SΓa] is not maximal Γ- ideal of S.

Then: i⟹ii⟹iii⟹ iv⟹ v.

Proof: i ⇒ ii By Lemma 3.3.
ii ⇒ iii⇒ iv Clear.
iv ⇒ v Let B be a S_Γ-subact of A and $a \in$ A with ann_S(B) ⊆ ann_S(SΓa). By hypothesis $a \in B$, thus SΓa ⊆ B and hence SΓA ⊆ B. Since B ⊆ $S\Gamma$A. Therefore, [B $:_S$ SΓa]=S which implies [B $:_S$ SΓa] is not maximal Γ-ideal.

Corollary 3.24. Let S be a Γ-monoid and K, L be S_Γ-subacts of co-multiplication S_Γ-act A. Then K ⊆ L if and only if there exists a S_Γ-monomorphism $f: K \to L$. Moreover, K=L if and only if [θ :$_S$ K]= [θ :$_S$ L].

Proof: (⟹) Clear. (⟸) Let $f: K \to L$ be a S_Γ-monomorphism .Then K ≅ $f(K)$ and hence [θ :$_S$ K]=[θ :$_S$ $f(K)$] .By Theorem 3.23, K=$f(K)$⊆L

Lemma 3.25 Let K be a Γ-ideal of Γ–monoid S and A be a SΓ-act. If A is faithful multiplication, then K=[KΓA:A]
Proof: Let $k \in$ K. Then kΓA ⊆ KΓA. Hence, $k \in$ [KΓA:A].Conversely, suppose $s \in$ [KΓA:A] then s ΓA⊆ KΓA and hence for all $\alpha \in \Gamma$, $a \in A$ there exists $k \in K$, $\beta \in \Gamma$ and $a_\circ \in A$ such that $s\alpha a = k\beta a_\circ$ that is $s\alpha a = k\beta a_\circ$ for all $\alpha \in \Gamma$, $a \in A$. $In\ particular$ $s\beta a_\circ = k\beta a_\circ$. By faithfulness of A, we have $s = k$. So, $s \in$ K. Therefore, K= [KΓA:A].

Theorem 3.26. Let S be a Γ-monoid and A be a faithful multiplication and co-multiplication S_Γ-act, then the following statements hold.

i. for all S_Γ-subact B of A, [B:$_S$ A] is a co-multiplication Γ-ideal of S,
ii. A Γ-monoid S is a co-multiplication as an S_Γ-act.

Proof

(i) Let B be a S_Γ-subact of S_Γ-act A. Then there exists a Γ-ideal K of S such that B=[θ :$_A$ K], thus [B:$_S$ A]=[[θ :$_A$ K]:$_S$ A]= [θ :$_S$ KΓA] = ann$_S$(KΓA).

(ii) Let K be a Γ-ideal of S, put B=KΓA is S_Γ-subact of A. Since A is faithful multiplication S_Γ-act. Then for every S_Γ-subact of A and every Γ-ideal of S; [B:$_S$ A]=[KΓA :$_S$ A]=K (Lemma 3.25). By (i), K is a co-multiplication ideal.

Conclusion*:*
In this paper, a generalization of multiplication gamma acts was presented. In the module theory, this concept is studied in a different way from how it was in this paper. Some basic results and properties of this concept were discussed. Since every co-multiplication gamma acts is multiplication gamma acts, to achieve them, several additional conditions were introduced.

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