Generalized Hyers - Ulam - Rassias Stability of a Quadratic Functional Equation

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Abstract

In this paper, we investigate the generalized Hyers-Ulam-Rassias stability of a new quadratic functional equation

\[ f(2x+y) - 4f(x) + f(y) + f(x+y) - f(x-y) = 0 \]

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Introduction


The functional equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \] (1.1)
is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. A generalized Hyers – Ulam stability problem for the quadratic functional equation was proved by skof [22]. For mappings \( f : X \to Y \); where \( X \) is a normed space \( Y \) is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group. Czerwik [7] proved the generalized Hyer-Ulam-Rassias stability of the quadratic functional equation (1.1) and Park [17] proved the generalized Hyer-Ulam-Rassias stability of the quadratic functional equation in Banach modules over a C*-algebra. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (See [12, 14, 16, 17, 18, 21]).

In this paper, we discuss a new quadratic functional equation.

\[
f(2x + y) = 4f(x) + f(y) + f(x + y) - f(x - y)
\]  

(1.2)

The generalized Hyers – Ulam – Rassias stability of the equation (1.2) is dealt with here. As a result of the paper, we have a much better possible upper bound for (1.2) than S.Czerwik and Skof – Cholewa.

**Hyers – Ulam – Rassias Stability of (1.2)**

In this section, let \( X \) be a real vector space and let \( Y \) be a Banach space. We will investigate the Hyers – Ulam – Rassias stability problem for functional equation (1.2)

Let us Define

\[
Df(x, y) = f(2x + y) - 4f(x) - f(y) - f(x + y) + f(x - y)
\]

Now we shall recall some theorems which will be useful in proving our results.

**Theorem 2.1** ([7]). If a function \( f : X \to Y \), where \( X \) is an abelian group and \( Y \) a Banach space, satisfies the inequality

\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^q)
\]

for \( p \neq 2 \) and for all \( x, y \in X \), then there exists a unique quadratic function \( Q \) such that

\[
\|f(x) - Q(x)\| \leq \frac{\varepsilon \|x\|^p}{4 - 2^p} + \frac{\|f(0)\|}{3}
\]

for all \( x \in X \).

**Theorem 2.2** ([6]). If a function \( f : X \to Y \), where \( X \) is an abelian group and \( Y \) is a Banach space, satisfies the inequality.

\[
\|f(x + y) + f(x - y) - 2(x) - 2f(y)\| \leq \varepsilon
\]

for all \( x, y, \in X \), then there exists a unique quadratic function \( Q \) such that
\[ \|f(x) - Q(x)\| \leq \frac{\varepsilon}{2} \]

for all \( x \in X \) and for all \( x \in X - 0 \), and \( \|f(0)\| = 0 \)

**Theorem 2.3** Let \( \psi : X^2 \to \mathbb{R}^+ \) be a function such that
\[
\sum_{i=0}^{\infty} \psi \left( \frac{2^i x, 0}{4^i} \right) \text{ converges and } \lim_{n \to \infty} \frac{\psi(2^i x, 2^n y)}{4^n} \quad (2.1)
\]

for all \( x, y \in X \). If a function \( f : X \to Y \) satisfies
\[
\|Df(x, y)\| \leq \psi(x, y) \quad (2.2)
\]

for all \( x, y \in X \), then there exists one and only one quadratic function \( Q : X \to Y \) which satisfies (1.2) and the inequality
\[
\|f(x) - Q(x)\| \leq \frac{1}{8} \sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i} \quad (2.3)
\]

for all \( x \in X \). The function \( Q \) is defined by
\[
Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n} \quad (2.4)
\]

for all \( x \in X \).

**Proof**: Letting \( x = y = 0 \) in (1.2), we get \( f(0) = 0 \). Putting \( y = 0 \) in (2.2) and dividing by 8, we get
\[
\|f(x) - \frac{f(2x)}{4}\| \leq \frac{1}{8} \psi(x, 0) \quad (2.5)
\]

for all \( x \in X \). Replacing \( x \) by \( 2x \) in (2.5) and dividing by 4 and summing the resulting inequality with (2.5), we get
\[
\|f(x) - \frac{f(2x)}{4}\| \leq \frac{1}{8} \left[ \psi(x, 0) + \frac{\psi(2x, 0)}{4} \right] \quad (2.6)
\]

for all \( x \in X \). Using induction on a positive integer \( n \), we obtain that
\[
\|f(x) - \frac{f(2^n x)}{4^n}\| \leq \frac{1}{8} \sum_{i=0}^{n-1} \frac{\psi(2^i, 0)}{4^i} \leq \frac{1}{8} \sum_{i=0}^{\infty} \frac{\psi(2^i x, 0)}{4^i} \quad (2.7)
\]

for all \( x \in X \).

Now, for \( m, n > 0 \)
Since the right-hand side of the inequality (2.8) tends to 0 as \( n \) tends to infinity, the sequence \( \left\{ \frac{f(2^n x)}{4^n} \right\} \) is a cauchy sequence.

Therefore, we may define \( Q(x) = \lim_{n \to \infty} \left\{ \frac{f(2^n x)}{4^n} \right\} \) for all \( x \in X \). Letting \( n \to \infty \) in (2.7), we arrive at (2.3).

Next, we have to show that \( Q \) satisfies (1.2). Replacing \( x, y \) by \( 2^n x, 2^n y \) in (2.2) and dividing by \( 4^n \), it then follows that

\[
\frac{1}{4^n} \| f(2^n(2x+y)) - 4f(2^n x) - f(2^n y) - f(2^n(x+y)) + f(2^n(x-y)) \| \leq \frac{1}{4^n} \psi(2^n x, 2^n y)
\]

Taking limit as \( n \to \infty \), using (2.1) and (2.4), we see that

\[
\| Q(2x + y) - 4Q(x) - Q(y) - Q(x + y) + Q(x - y) \| \leq 0
\]

Which gives

\[
Q(2x + y) = 4Q(x) + Q(y) + Q(x + y) - Q(x - y)
\]

Therefore, we have that \( Q \) satisfies (2.1) for all \( x, y \in X \). To prove the uniqueness of the quadratic function \( Q \), let us assume that there exists a quadratic function \( Q^1 : X \to Y \) which satisfies (1.2) and the inequality (2.3). But we have \( Q(2^n x) = 4^n Q(x) \) and \( Q^1(2^n x) = 4^n Q^1(x) \) for all \( x \in X \) and \( n \in N \). Hence it follows from (2.3) that

\[
\| Q(x) - Q^1(x) \| = \frac{1}{4^n} \| Q(2^n x) - Q^1(2^n x) \|
\]

\[
\leq \frac{1}{4^n} (\| Q(2^n x) - f(2^n x) \| + \| f(2^n x) - Q^1(2^n x) \|)
\]

\[
\leq \frac{1}{4^n} \sum_{i=0}^{\infty} \psi(2^{i+n}, 0) \to 0 \text{ as } n \to \infty
\]
Therefore Q is unique. This completes the proof of the theorem.

From Theorem 2.1, we obtain the following corollaries concerning the stability of the equation (1.2).

**Corollary 2.4:** Let \( X \) be a real normed space and \( Y \) a Banach space. Let \( \epsilon, p, q \) be real numbers such that \( \epsilon \geq 0, q > 0 \) and either \( p, q < 2 \), or \( p, q > 2 \). Suppose that a function \( f : X \to Y \) satisfies

\[
\| D f(x, y) \| \leq \epsilon (\| x \|^p + \| y \|^q)
\]

for all \( x, y \in X \). Then there exists one and only one quadratic function \( Q : X \to Y \) which satisfies (1.2) and the inequality.

\[
\| f(x) - Q(x) \| \leq \frac{\epsilon}{2\| 4 - 2^p \|} \| x \|^p
\]

for all \( x \in X \). The function \( Q \) is defined in (2.4). Furthermore, if \( f(t x) \) is continuous for all \( t \in \mathbb{R} \) and \( x \in X \), then \( f(tx) = t^2 f(x) \).

**Proof:** Taking \( \psi(x, y) = \epsilon (\| x \|^p + \| y \|^q) \) and applying Theorem 2.1, the equation (2.3) give rise to equation (2.10) which proves corollary 2.4.

**Corollary 2.5** Let \( X \) be a real normed space and \( Y \) be a Banach space. Let \( \epsilon \) be real number. If a function \( f : X \to Y \) satisfies

\[
\| Df(x, y) \| \leq \epsilon
\]

for all \( x, y \in X \), then there exists one and only one quadratic function \( Q : X \to Y \) which satisfies (1.2) and the inequality.

\[
\| f(x) - Q(x) \| \leq \frac{\epsilon}{4}
\]

for all \( x \in X \). The function \( Q \) is defined in (2.4). Furthermore, if \( f(t x) \) is continuous for all \( t \in \mathbb{R} \) and \( x \in X \) then, \( f(tx) = t^2 f(x) \).

**References**


