Intuitionistic Fuzzy Linear Transformations

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Abstract

In this paper, we introduce the concept of linear transformation on intuitionistic fuzzy vector space and prove that the set of all linear transformations between two intuitionistic fuzzy vector spaces forms a vector space under the intuitionistic fuzzy operations. By using the standard intuitionistic fuzzy linear combination of a vector, one to one correspondence between the set of all linear transformations on a finitely generated subspace of \( \mathbb{V}_n \) and the set of all intuitionistic fuzzy matrices \((\text{IF})_n\) is obtained.

Keywords: Fuzzy matrix, Intuitionistic fuzzy matrix, Intuitionistic fuzzy vector space, Intuitionistic fuzzy linear transformation, Standard basis.

Introduction

We deal with fuzzy matrices that is, matrices over the fuzzy algebra \( \mathcal{F}^M \) and \( \mathcal{F}^N \) with support \([0,1]\) and fuzzy operations \(\{+,,\}\) defined as \(a + b = \max\{a, b\}\), \(a.b = \min\{a, b\}\) for all \(a, b \in \mathcal{F}^M\) and \(a + b = \min\{a, b\}, a.b = \max\{a, b\}\) for all \(a, b \in \mathcal{F}^N\). Let \( \mathcal{F}^M_{mn} \) be the set of all \(mn\) Fuzzy matrices over \( \mathcal{F} \). A matrix \( A \in \mathcal{F}^M_{mn} \) is said to be regular if there exists \( X \in \mathcal{F}^M_{nm} \) such that \( AXA = A \), \( X \) is called a generalized inverse (g-inverse) of \( A \). In [4], Kim and Roush have developed the theory of fuzzy matrices, under max min composition analogous to that of Boolean matrices. Cho [3] has discussed the consistency of fuzzy matrix equations, if \( A \) is regular with a g-inverse \( X \), then \( b.X \) is a solution of \( x.A = b \). Further every invertible matrix is regular. For more details on fuzzy matrices one may refer [5], In[6], Meenakshi and inbam have introduced the concept of linear transformation on fuzzy vector spaces. The new concept of fuzzy linear transformation on vector spaces and fuzzy vector spaces are discussed in [11]. Atanassov has introduced and developed the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [1,2]. If \( A = (a_{ij}) \in (\text{IF})_{mn} \),
then \( A = \left( \langle a_{ij}, b_{ij} \rangle \right) \), where \( a_{ij} \) and \( b_{ij} \) are the membership values and non-membership values of \( a_{ij} \) in \( A \) respectively with respect to the fuzzy sets \( \mu \) and \( \nu \), maintaining the condition \( 0 \leq a_{ij} + b_{ij} \leq 1 \). In [7], we have studied the structure of row space and column space of intuitionistic fuzzy matrices. The maximum and minimum solutions of fuzzy relational equations involving membership and non-membership matrix of the intuitionistic fuzzy matrix are determined in [8]. In [9], we prove that any finitely generated subspace of \( V_n \) over the intuitionistic fuzzy algebra (IF) has a unique standard basis. The concept of semiring of intuitionistic fuzzy matrices are studied in [10].

In this paper, we define a linear transformation and prove that set of all linear transformations between two intuitionistic fuzzy vector spaces forms a vector space.

**Preliminaries**

Let \( (IF)_{mxn} \) be the set of all intuitionistic fuzzy matrices of order \( mxn \) Let \( (IF)_{mxn} \) be the set of all intuitionistic fuzzy matrices of order \( mxn \). First we shall represent \( A \in (IF)_{mxn} \) as Cartesian product of fuzzy matrices. The Cartesian product of any two matrices \( A=\left( a_{ij} \right)_{mxn} \) and \( B=\left( b_{ij} \right)_{mxn} \), denoted as \( \langle A, B \rangle \) is defined as the matrix whose \( ij^{th} \) entry is the ordered pair \( \langle A, B \rangle = \left( \langle a_{ij}, b_{ij} \rangle \right) \). For \( A = \left( a_{ij} \right)_{mxn} = \left( \langle a_{ij}, a_{ij} \rangle \right) \). We define \( A_{\mu} = \left( a_{ij} \right)_{mxn} \) as the membership part of \( A \) and \( A_{\nu} = \left( a_{ij} \right)_{mxn} \) as the non-membership part of \( A \). Thus \( A = \langle A_{\mu}, A_{\nu} \rangle \) with \( A_{\mu} \in F_{mxn}^M \), \( A_{\nu} \in F_{mxn}^N \).

We shall follow the matrix operations on intuitionistic fuzzy matrices as defined in our earlier work [7].

For \( A, B \in F_{mxn}^M \), if \( A = \left( a_{ij} \right) \) and \( B = \left( b_{ij} \right) \) then

\[
A+B = \left( \max \{a_{ij}, b_{ij} \} \right)
\]

\[
A B = \left( \max \min \{a_{ik}, b_{kj} \} \right)
\]

The \( nxn \) identity matrix \( I_n^M \) in \( F_n^M \) is the matrix defined as \( \left( I_n^M \right)_{ij} = 1 \) if \( i=j \) and \( \left( I_n^M \right)_{ij} = 0 \) if \( i \neq j \) and the \( nxn \) zero matrix \( O \) is the matrix all of whose entries are zero satisfying the following properties:

\[
A I_n^M = I_n^M A = A, \text{ for all } A \in F_n^M
\]

\[
O A = A O = O.
\]
For $A, B \in \mathcal{F}^N_{mn}$, if $A = (a_{ij})$ and $B = (b_{ij})$ then

\begin{align}
A + B &= \left( \min_k \{a_{ij}, b_{ij}\} \right) \\
A \cdot B &= \left( \min_k \{a_{ik}, b_{kj}\} \right)
\end{align}

The $n \times n$ identity matrix $I^N_n$ in $\mathcal{F}^N_n$ is the matrix defined as $I^N_n = 0$ if $i = j$ and $I^N_n = 1$ if $i \neq j$. The $n \times n$ zero matrix $J$ in $\mathcal{F}^N_n$ is the matrix all of whose entries are 1 satisfying the following properties:

\begin{align}
A I^N_n &= I^N_n A = A, \text{ for all } A \in \mathcal{F}^N_n \\
A J &= J A = J.
\end{align}

For $A, B \in (IF)_{mn}$, if $A = \left( a_{ij}, a_{ijv} \right)$ and $B = \left( b_{ij}, b_{ijv} \right)$ then

\begin{align}
A + B &= \left( \max_k \{a_{ij}, b_{ij}\}, \min_k \{a_{ijv}, b_{ijv}\} \right) \\
A \cdot B &= \left( \min_k \{a_{ik}, b_{kj}\}, \max_k \{a_{ikv}, b_{kjv}\} \right)
\end{align}

Let us define the order relation on $(IF)_{mn}$ as,

\begin{align}
A \preceq B \iff a_{ij} \leq b_{ij} \text{ and } a_{ijv} \geq b_{ijv}, \text{ for all } i \text{ and } j
\end{align}

**Definition 2.1**\[^7\]

An $A \in (IF)_{mn}$ is said to be regular if there exists $X \in (IF)_{nm}$ satisfying $AXA = A$ and $X$ is called a generalized inverse (g-inverse) of $A$ which is denoted by $A$. Let $A\{1\}$ be the set of all g-inverses of $A$.

**Definition 2.2**\[^10\]

A square intuitionistic fuzzy matrix is called intuitionistic fuzzy permutation matrix, if every row and column contains exactly one $<1,0>$ and all other entries are $<0,1>$. Let $P_n$ be the set of all $n \times n$ permutation matrix in $(IF)_n$.

**Lemma 2.1**\[^9\]

Any two basis for a finitely generated subspace of the intuitionistic fuzzy algebra $(\mathcal{F}) = < \mathcal{F}^M, \mathcal{F}^N >$ have the same cardinality and any finitely generated subspace over $(\mathcal{F})$ has a unique standard basis.

**Lemma 2.2**\[^9\]

Let $S$ be a finitely generated subspace of $V_n$ and let $c = \{c_1, c_2, \ldots, c_n\}$ be the standard basis for $S$. Then any vector $x \in S$ can be expressed uniquely as a linear combination of the standard basis.
Lemma 2.3[9]
Let \{c_1,c_2,\ldots,c_n\} be the standard basis of the subspace W in V_n. In the standard fuzzy linear combination of the basis vector c_i, the i^{th} coefficient of c_i is &lt;1,0&gt;.

Intuitionistic Fuzzy Linear Transformations (IFLT)
In this section, we introduce the concept of linear transformation on intuitionistic fuzzy vector spaces and prove that the set of all linear transformations between two intuitionistic fuzzy vector spaces forms a vector space under the intuitionistic fuzzy operations.

Definition 3.1
Let V and V' be vector spaces over the intuitionistic fuzzy algebra(ƗƑ). A mapping T of V into V' is called a intuitionistic fuzzy linear transformation if for any x, y ∈ V and α ∈ (ƗƑ).

1. \(T(x+y) = T(x) + T(y)\)
2. \(T(\alpha x) = \alpha T(x)\)

Theorem 3.1
Let V and V' be vector spaces over the intuitionistic fuzzy algebra(ƗƑ). Then L(V,V'), the set of all intuitionistic fuzzy linear transformation from V into V' is a intuitionistic fuzzy vector space over the intuitionistic fuzzy algebra(ƗƑ) under addition and multiplication defined by, \((T_1+T_2)x = T_1(x) + T_2(x)\) and \((\alpha T_1) x = \alpha T_1(x)\) for all \(T_1, T_2 \in L(V,V')\), \(x \in V\) and \(\alpha \in (ƗƑ)\).

Proof
Let L(V,V') is closed with respect to ‘+’ and ‘.’

For any \(x,y \in V\).
\((T_1+T_2)(x+y) = T_1(x+y) + T_2(x+y)\)
\(= T_1(x) + T_1(y) + T_2(x) + T_2(y)\)
\(= (T_1+T_2)x + (T_1+T_2)y\) for every \(T_1,T_2 \in L(V,V')\)

For \(x \in V\), \((T_1+T_2)(\alpha x) = T_1(\alpha x) + T_2(\alpha x)\)
\(= \alpha T_1(x) + \alpha T_2(x)\)
\(= \alpha [T_1(x) + T_2(x)]\)
\(= \alpha [(T_1+T_2)(x)]\) for every \(T_1,T_2 \in L(V,V')\) and \(\alpha \in (ƗƑ)\).

Thus \(T_1+T_2 \in L(V,V')\) for all \(T_1,T_2 \in L(V,V')\).
For $\alpha \in (\mathcal{IF})$ and $T \in L(V,V^{'})$.

\[(\alpha T)(x+y) = \alpha [T(x+y)]
= \alpha [T(x)+ T(y)]
= \alpha T(x)+ \alpha T(y).
\]

\[(\alpha T)(\beta x) = \alpha [T(\beta x)]
= \alpha [\beta T(x)]
= \beta [(\alpha T)(x)].\]

Hence $\alpha T \in L(V,V^{'})$.

For $T_1, T_2, T_3 \in L(V,V^{'})$ and $\alpha, \beta \in (\mathcal{IF})$ the following hold:

i. $T_1 + T_2 = T_2 + T_1$

ii. $(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$

iii. $(\alpha \beta) T_1 = \alpha (\beta T_1)$

iv. $(\alpha + \beta) T_1 = \alpha T_1 + \beta T_1$

v. $\alpha (T_1 + T_2) = \alpha T_1 + \alpha T_2$

vi. $\beta (T_1 + T_2) = \alpha (T_1 + \beta T_2)$

vii. $T_1 + 0 = 0 + T_1 = T_1$

viii. $0 T_1 = T_1 0 = 0$

Proof of (i) to (viii) can be easily verified. Hence $L(V,V^{'})$ is a intuitionistic fuzzy vector space over $(\mathcal{IF})$.

Theorem 3.2

For an intuitionistic fuzzy vector space $V$ over $(\mathcal{IF})$, $L(V)$ is an algebra under multiplication defined by $T_1 T_2 (x) = T_1 (T_2 (x))$ for all $T_1, T_2 \in L(V)$.

Proof

$L(V)$ is an intuitionistic fuzzy vector space follows from theorem(3.1). We will prove that $T_1 T_2 \in L(V)$, for $T_1, T_2 \in L(V)$.

\[(T_1 T_2) (x+y) = T_1 (T_2 (x+y))
= T_1 (T_2 (x)+ T_2 (y))
= T_1 (T_2 (x)) + T_1 (T_2 (y))
= (T_1 T_2) (x) + (T_1 T_2) (y).
\]

\[(T_1 T_2) (\alpha x) = T_1 (T_2 (\alpha x))
= T_1 (\alpha T_2 (x)).\]
\[ \alpha T_1 (T_2(x)) = (T_1 T_2)(x). \]

Thus \( T_1 T_2 \in L(V) \). Hence the proof. \( \square \)

**Definition 3.2**

Let \( L \) and \( L' \) be vector spaces over the intuitionistic fuzzy algebra \((\tilde{I}_F)\). \( L \) and \( L' \) are said to be intuitionistic fuzzy isomorphic if there exists a one to one and onto linear transformation between them, that is, \( f \) is one to one, onto map satisfying

i. \( f(x+y) = f(x) + f(y), \) for \( x, y \in L \)

ii. \( f(\alpha x) = \alpha f(x), \) for \( \alpha \in (\tilde{I}_F) \).

**Theorem 3.3**

Let \( L \) and \( L' \) be finite dimensional vector spaces over the intuitionistic fuzzy algebra \((\tilde{I}_F)\). \( L \) and \( L' \) are said to be intuitionistic fuzzy isomorphic if and only if \( \dim(L) = \dim(L') \).

**Proof**

Let \( f : L \rightarrow L' \) be a fuzzy isomorphism. If \( \{x_1, x_2, \ldots, x_k\} \) is a standard basis of \( L \).

Then it can be verified that \( \{f(x_1), f(x_2), \ldots, f(x_k)\} \) is a standard basis of \( L' \). Hence \( \dim(L) = \dim(L') \).

Conversely, \( \dim(L) = \dim(L') \), then \( L \) and \( L' \) will have same cardinality. Let \( \{x_1, x_2, \ldots, x_k\} \) and \( \{y_1, y_2, \ldots, y_k\} \) be standard basis of \( L \) and \( L' \) respectively. By lemma (2.2), any \( x \in L \) has a unique standard linear combination of basis vectors of the form \( x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_k x_k \). Define \( f(x) = y \), where \( y = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_k y_k \). It can be verified that \( f \) satisfies the definition of a intuitionistic fuzzy isomorphism. Hence \( L \) and \( L' \) are intuitionistic fuzzy isomorphic. \( \square \)

**Intuitionistic fuzzy matrices associated with IFLT**

Let \( B = \{c_1, c_2, \ldots, c_n\} \) be an ordered standard basis for a subspace \( W \) of intuitionistic fuzzy vector space \( V_n \). Then by lemma(2.2), each vector in \( W = \langle W_{\mu}, W_{\nu} \rangle \) is uniquely expressible as a standard fuzzy linear combination of \( c_i \)'s. For each \( i, c_i = \langle c_{i\mu}, c_{i\nu} \rangle \), where \( c_{i\mu} \in F^\mu \) and \( c_{i\nu} \in F^\nu \) and \( B_{\mu} = \{c_1_{\mu}, c_2_{\mu}, \ldots, c_n_{\mu}\} \) and \( B_{\nu} = \{c_1_{\nu}, c_2_{\nu}, \ldots, c_n_{\nu}\} \) are the standard basis for \( W_{\mu} \) and \( W_{\nu} \) respectively.

If \( T \) is a linear transformation on \( W \) to itself, then for each \( c_j = \langle c_{j\mu}, c_{j\nu} \rangle \) the vector \( T(c_j) = T(< c_{j\mu}, c_{j\nu}>) \) is in \( W \).

Let \( T(c_j) = T(< c_{j\mu}, c_{j\nu}>) = \sum_{i=1}^{n} \alpha_{ij\mu} c_{i\mu} + \alpha_{ij\nu} c_{i\nu} \) be the standard fuzzy linear combination in terms of the standard basis vectors.
Construct the matrix \([T]=<[T_\mu], [T_\nu]>\), where \([T_\mu] \in \mathcal{F}^M\) and \([T_\nu] \in \mathcal{F}^N\), whose \(j^{th}\) column is the transpose of the intuitionistic row vector \(<a_{1j\mu}, a_{1j\nu}>, <a_{2j\mu}, a_{2j\nu}>, \ldots, <a_{nj\mu}, a_{nj\nu}>>\). Since \(a_{ij} = <a_{ij\mu}, a_{ij\nu}>'s\) are uniquely determined by \(T(c_j) = T(<c_{ij\mu}, c_{ij\nu}>)\) for each \(j\), \([T]\) is the unique intuitionistic fuzzy matrix corresponding to the linear transformation \(T\) on \(L(W)\) with respect to the unique standard ordered basis \(B\) of the subspace \(W\). Let us denote it as \([T]_B'.\)

**Theorem 4.1**
Let \(B = \{c_1, c_2, \ldots, c_n\}\) be an ordered standard basis for a subspace \(W\) of \(V_n\). Then the mapping \(T \rightarrow T'\), which assigns to each linear transformation \(T\) its matrix relative to \(B\) is an isomorphism of the algebra \(L(W)\) onto the matrix algebra \((\mathcal{I}\mathcal{F})_n\).  

**Proof**  
By using the unique standard basis \(B = \{c_1, c_2, \ldots, c_n\}\) we have assigned a matrix  
\([T] = [a_{ij\mu}, a_{ij\nu}]\) to each linear transformation \(T\) on \(W\).  

**Claim:** \(T \rightarrow [T]\) is one to one  
Let \([T_1] = [a_{ij\mu}, a_{ij\nu}]\) and \([T_2] = [\gamma_{ij\mu}, \gamma_{ij\nu}]\). Then  
\[T_1(c_j) = T_1(<c_{ij\mu}, c_{ij\nu}>) = \sum_{i=1}^{n} <a_{ij\mu}, a_{ij\nu}> <c_{ij\mu}, c_{ij\nu}> \]  
\[T_2(c_j) = T_2(<c_{ij\mu}, c_{ij\nu}>) = \sum_{i=1}^{n} <\gamma_{ij\mu}, \gamma_{ij\nu}> <c_{ij\mu}, c_{ij\nu}>.\]

Let \(x = \sum_{j=1}^{n} \beta_j <c_{ij\mu}, c_{ij\nu}>\) be the standard fuzzy linear combination of \(x\) in terms of the standard basis vectors \(\{c_1, c_2, \ldots, c_n\}\).  
Suppose \([T_1] = [T_2]\) then \(a_{ij\mu} = \gamma_{ij\mu}\) and \(a_{ij\nu} = \gamma_{ij\nu}\) for all \(i, j = 1, 2, \ldots, n\).  
\[T_1(x) = T_1[\sum_{j=1}^{n} \beta_j <c_{ij\mu}, c_{ij\nu}>] = \sum_{j=1}^{n} \beta_j T_1(<c_{ij\mu}, c_{ij\nu}>)\]  
\[= \sum_{j=1}^{n} \beta_j \left(\sum_{i=1}^{n} <a_{ij\mu}, a_{ij\nu}> <c_{ij\mu}, c_{ij\nu}>\right)\]  
\[= \sum_{j=1}^{n} \beta_j \left(\sum_{i=1}^{n} <\gamma_{ij\mu}, \gamma_{ij\nu}> <c_{ij\mu}, c_{ij\nu}>\right)\]  
\[= \sum_{j=1}^{n} \beta_j T_2(<c_{ij\mu}, c_{ij\nu}>).\]
\[ T_2 \sum_{j=1}^{n} \beta_j < c_{i_ju}, c_{j_v}> = T_2(x) \]

Hence \( T \to [T] \) is one to one.

**Claim:** \( T \to [T] \) is onto.

Let \([ T ] = [ < \alpha_{iju}, \alpha_{ijv}> ]\) be an element in \((\mathbb{F}_h)\). Now define a map \( T \) on the standard basis vectors \( B = \{c_1, c_2, \ldots, c_n\} \) as

\[
T(x) = T \left[ \sum_{j=1}^{n} \beta_j < c_{i_ju}, c_{j_v}> \right] = \sum_{j=1}^{n} \beta_j T(< c_{i_ju}, c_{j_v}>)
\]

\[
= \sum_{i,j=1}^{n} \beta_j \left( \sum_{i=1}^{n} < \alpha_{iju}, \alpha_{ijv}> < c_{i_ju}, c_{j_v}> \right)
\]

\[
= \sum_{i,j=1}^{n} \beta_j \left( \alpha_{iju}, \alpha_{ijv}> \right) < c_{i_ju}, c_{j_v}> \text{ is well defined.}
\]

We can show that \( T \) is a linear transformation on \( W \).

Let \( x = \sum_{j=1}^{n} \beta_j < c_{i_ju}, c_{j_v}> \) and \( y = \sum_{j=1}^{n} \gamma_j < c_{i_ju}, c_{j_v}> \). Then \( x+y = \sum_{j=1}^{n} (\beta_j + \gamma_j) < c_{i_ju}, c_{j_v}> \).

(4.1) \[
T(x+y) = \sum_{i,j=1}^{n} [(\beta_j + \gamma_j) < \alpha_{iju}, \alpha_{ijv}>] < c_{i_ju}, c_{j_v}>
\]

(4.2) \[
T(x) + T(y) = \sum_{i,j=1}^{n} [\alpha_{iju}, \alpha_{ijv}> (\beta_j + \gamma_j)] < c_{i_ju}, c_{j_v}>
\]

From (4.1) and (4.2) it follows that, \( T(x+y) = T(x) + T(y) \).

\[
T(\alpha x) = T(\sum_{j=1}^{n} \alpha \beta_j < c_{i_ju}, c_{j_v}>) = \sum_{i,j=1}^{n} (\alpha \beta_j) < c_{i_ju}, c_{j_v}>.
\]

\[
= \alpha \sum_{i,j=1}^{n} (\alpha_{iju}, \alpha_{ijv}> \beta_j) < c_{i_ju}, c_{j_v}> = \alpha T(x)
\]

Thus \( T \) is a linear transformation on \( W \).

Now, \( (T_1 + T_2) < c_{i_ju}, c_{j_v}> = (T_1 < c_{i_ju}, c_{j_v}> + T_2 < c_{i_ju}, c_{j_v}>) \)

\[
= \sum_{i=1}^{n} < \alpha_{iju}, \alpha_{ijv}> < c_{i_ju}, c_{j_v}> + \sum_{i=1}^{n} < \gamma_{iju}, \gamma_{ijv}> < c_{i_ju}, c_{j_v}>
\]

\[
= \sum_{i=1}^{n} ( < \alpha_{iju}, \alpha_{ijv}> + < \gamma_{iju}, \gamma_{ijv}> ) < c_{i_ju}, c_{j_v}>.
\]

If we define addition in \((\mathbb{F}_h)\) by \([< \alpha_{iju}, \alpha_{ijv}>] + [< \beta_{iju}, \beta_{ijv}>] = \max \{ \alpha_{iju}, \beta_{ijv} \} \),
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\[ \beta_{ij}, \min\{\alpha_{ijv}, \beta_{ij}\} \] and \[ [\alpha < \alpha_{ij}, \alpha_{ijv}>] = [\min\{\alpha, \alpha_{ij}\}, \max\{1- \alpha, \alpha_{ijv}\}] \].

(4.3) \[ [T_1 + T_2] = [T_1] + [T_2] \] and

(4.4) \[ [\alpha T_1] = \alpha [T_1], \] for any \( \alpha \in \mathcal{I}_\mathcal{F}_n \).

Finally, \( (T_1 T_2) c_j = (T_1 T_2) < c_{jv}, c_{jv} > = T_1 (T_2 < c_{jv}, c_{jv} >) \)

\[ = T_1 \sum_{k=1}^{n} < \beta_{kju}, \beta_{kjv} > c_{kj}, c_{kjv} >. \]

\[ = \sum_{k=1}^{n} < \beta_{kju}, \beta_{kjv} > \left( T_1 < c_{kj}, c_{kjv} > \right). \]

\[ = \sum_{k=1}^{n} < \beta_{kju}, \beta_{kjv} > \left( \sum_{i=1}^{n} < \alpha_{ikj}, \alpha_{ikv} > < c_{ki}, c_{kv} > \right). \]

\[ [< \alpha_{ij}, \alpha_{ijv}>] [< \beta_{ij}, \beta_{ijv}>] = \sum_{i=1}^{n} < \alpha_{ikj}, \alpha_{ikv} > < \beta_{kju}, \beta_{kjv} > < c_{ki}, c_{kv} >. \]

(4.5) Therefore \[ [T_1] [T_2] = [T_1 T_2] \].

The operations defined by (4.3),(4.4) and (4.5) are the standard max-min,min-max operations for intuitionistic fuzzy matrices.

**Remark 4.1**

The image of the zero linear transformation under the mapping \( T \rightarrow [T] \) is the zero matrix, all of whose entries are \(<0,1>\).

**Remark 4.2**

In particular for the identity transformation on \( W \), since \( I(c_i) = c_i \) for each basis vector \( c_i \) in the standard basis \( B=\{c_1,c_2,\ldots,c_n\} \) of \( W \). From lemma(2.3),the diagonal entries of \( I \) is \(<1,0>\).

**Example 4.1**

Let \( W \) be the subspace of \( W_3 \).The identity mapping \( I: W \rightarrow W \).Since \( I(x) = x \) for all \( x \in W \), the standard fuzzy linear combination of the standard basis vector

\[ (<0.5,0>,<0.5,0.5>,<0.5,0>) \] is obtained in example(3.2)[9] as,

\[ ( <0.5,0>,<0.5,0.5>,<0.5,0> ) = ( <1,0> ( <0.5,0>,<0.5,0.5>,<0.5,0> ) + <0.5,0.5> ( <0.1>,<1,0>,<0.5,0.5> ) + <0.5,0> ( <0.1>,<0.5,0.5>,<1,0> ). \]

Similarly, the standard fuzzy linear combination of the other basis vectors are obtained as,

\[ ( <0.5,0> (<0.5,0>,<0.5,0.5>,<0.5,0,0>) + <1,0> ( <0.1>,<1,0>,<0.5,0.5> ) + <0.5,0> (<0.1>,<0.5,0.5>,<1,0> ) \) and

\[ ( <0.1>,<0.5,0.5>,<1,0> ) = ( <0,1> ( <0.5,0>,<0.5,0.5>,<0.5,0> ) + <0.5,0.5> ( <0.1>,<1,0>,<0.5,0.5> ) + <0.5,0> ( <0.1>,<0.5,0.5>,<1,0> ). \]
Hence the matrix \([I]\) corresponding to the identity linear transformation with respect to the standard basis \(\{(<0.5,0>,<0.5,0.5>,<0.5,0>),(<0,1>,<1,0>,<0.5,0.5>), (<0.5,0.5>, <1,0>) \} \) is,

\[
[I]_B = \begin{pmatrix} <1,0> & <0,0.5> & <0,1> \\ <0.5,0.5> & <1,0> & <0.5,0.5> \\ <0.5,0> & <0,0.5> & <1,0> \end{pmatrix}
\]

**Corollary 4.1**

For any \(T \in L(W)\) and the identity transformation \(I \in L(W)\) the corresponding matrices \([T]\) and \([I]\) satisfy \([T] \cdot [I] = [I] \cdot [T] = [T] \) under the max-min, min-max compositions of intuitionistic fuzzy matrices.

**Proof**

Since \(T \cdot I = I = T\), by using (4.5) we get the matrix identity \([T] \cdot [I] = [I] \cdot [T] = [T] \).

\[\blacksquare\]

**References**