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A Computational Method for Solving Singular Perturbation Problems without First Derivative Term

K. Selvakumar

Department of Mathematics, Anna University of Technology Tirunelveli,
Tirunelveli—627 007, Tamil Nadu, India.
E-mail address: k_selvakumar10@yahoo.com

Abstract

A computational method is presented for solving singularly perturbed two point boundary value problems without a first derivative term. The absence of first derivative term leads to the boundary layer regions nearer the end points of the interval(both left and right points of the interval). The zeroth order asymptotic expansion is used to obtain the terminal boundary conditions. Then, the two boundary layer regions and one non-boundary layer region are created. And so, the given problem is split into three two-point boundary value problems. All these problems are efficiently solved by an uniform and optimal exponentially fitted finite difference scheme. Error estimates for the computational method is derived using maximum principle,. Numerical results are given in this paper to demonstrate the applicability of the computational method.

Keywords: singular perturbation problems, exponentially fitted, uniformly convergent, asymptotic expansion, finite difference schemes.

AMS (MOS) subject classification: 65F05, 65N30, 65N35, 650Y05.

Introduction

The numerical treatment for singular perturbation problems have always been far from trivial, because of the boundary layer behavior of the solution. These problems occur frequently in fluid mechanics, elasticity and other branches of applied mathematics, science and engineering. A few notable problems are boundary layer problems, WKB problems, convective heat transport problems with large Peclet number, etc. The area of singular perturbations is a field of increasing interest to applied mathematicians. To be specific, we consider the following singular perturbation problem(SPP):

$$Lu(x) \equiv - \varepsilon\, u''(x) + b(x)\, u(x) = f(x), \qquad (1a)$$

for 0 < x < 1 with

$$u(0) = \varphi_1 \text{ and } u(1) = \varphi_2 \qquad (1b)$$

where ε is a small parameter ($0 < \varepsilon << 1$), φ_1 , φ_2 are given constants, b(x) and f(x) are assumed to be sufficiently continuously differentiable functions in [0, 1], and $b(x) \geq \beta > 0$ on [0, 1], where β is some positive constant. Under these assumptions SPP(1a,b) has a unique solution u(x) which, in general, displays a boundary layer of width $O(\sqrt{\varepsilon})$ at x=0 and x= 1 for small values of ε [1-4, 811,14,15].

Uniformly convergent finite difference schemes for the SPP(1ab) have been examined by various authors [2-4, 9, 11, 14, 15]. All these schemes use constant mesh size and it is impractical if one wants to find local behavior of the solution in the neighborhood of ε , where ε is small.

Pearson[12] was perhaps the first to attempt something like net adjustments in finite difference schemes for the boundary value problem with first derivative term . Roberts[13] proposed a boundary value technique and introduced the idea of inner and outer region problems for the domain [0, 1]. Such type of technique is also discussed in [5-7]. Other works include Bender[1], Neyfeh]8] and O'Malley[10].

The objective of the paper is to present a new approach for solving the SPP(1a,b) . It is based on the asymptotic behavior of the solution of the SPP(1a,b). The method consists of the following ste3ps:

- the original problem is divided into left boundary layer region problem, the outer region problem and right boundary layer region problem
- terminal boundary conditions are obtained from the zeroth order asymptotic expansion for the solution of the SPP(1a,b)
- then, the new inner region problems are created and solved numerically using uniform and optimal exponentially fitted schemes with variable mesh
- in urn , the outer region problem is created and solved numerically using uniform and optimal exponentially fitted schemes which solve the reduced problem exactly for small values of ε with constant mesh
- finally, we combine the solutions of left boundary layer region problem, the outer region problem and right boundary layer region problem.

The process is to be repeated for different choices of the terminal points until the profiles stabilize in left boundary layer region and outer region and in , the outer region and the right boundary layer region.

In section 2 the description of the computational method is given. The error estimate of the solutions of three region problems are derived in section 3. The error estimate of the numerical solution of three region problems are derived in section 4 . The error estimate of the solutions of the computational method with respect to the solutions of three region problems are derived in section 5 . The numerical experimental results are presented in section 6.

Throughout this paper , we shall use the following notations

$b_0 = \sqrt{b(0)/\varepsilon}$,
$b_1 = \sqrt{b(1)/\varepsilon}$,
$x_p = t_p \sqrt{\varepsilon}$,
$x_q = (1 - x_p) \sqrt{\varepsilon}$,
$D_+ D_- u_i = (u_{i+1} - 2 u_i + u_{i-1}) / h^2$,
$\rho_1 = h_1 / \sqrt{\varepsilon}$,
$\rho_2 = h_2 / \sqrt{\varepsilon}$,
$\sigma (\rho_1) = \sigma (\rho_1 \sqrt{b(x_i)}) \sigma (- \rho_1 \sqrt{b(x_i)})$,
$\sigma (\rho_2) = \sigma (\rho_2 \sqrt{b(x_i)}) \sigma (- \rho_2 \sqrt{b(x_i)})$

where σ (x) and σ (- x) are Bernoulli's generating functions defined as

$\sigma (- x) = x/[1 - \exp(- x)]$ and $\sigma (x) = \exp(- x) \sigma (- x)$

for x>0 and C is independent of I, h_1, h_2 cad □ .

Computational method

Consider the SPP(1a,b). as the original problem. We split the original problem into three problems, namely, the left boundary layer region problem, the outer region problem and the right boundary layer region problem

First, we compute the left and right terminal boundary conditions as follows.

Left terminal boundary condition

Let x_p be the terminal point or common point or width or thickness of the left boundary layer region. To find the terminal boundary condition we use the solution of the reduced problem

$$b(x)\, u_0(x) = f(x),\ x \in (0, 1) \quad (2)$$

and the transformed equation

$$-- d^2 v_0(\tau)/d\tau^2 + b(0)\, v_0(\tau) = 0, \quad (3a)$$

$$-- d^2 w_0(\eta)/d\eta^2 + b(1)\, w_0(\eta) = 0,\ \tau ,\ \eta \in (0,\infty), \quad (3b)$$

$$v_0(\tau = 0) + w_0(\eta = 1/\sqrt{\varepsilon}) = \varphi_1 - u_0(0), \quad (3c)$$

$$v_0(\tau = 1/\sqrt{\varepsilon}) + w_0(\eta = 0) = \varphi_2 - u_0(1), \quad (3d)$$

The reduced problem (2) is got by setting □ = 0 in the SPP(1a,b). And the boundary value problems (3a,d) are obtained by the Taylor's expansion of the coefficients b(x) and f(x) about x = 0 and x = 1, making a change of variables

$$X \rightarrow \tau = x / \sqrt{\varepsilon} \text{ and } \eta = (1 - x) / \sqrt{\varepsilon}$$

and equating powers of □.

The zeroth order asymptotic expansion for the solution of the SPP(1a, b) is given by

$$U = u_0 + v_0 + w_0 \ . \tag{4}$$

where

$$u_0(x) = f(x) / b(x) , \tag{5}$$

$$v_0(x) = (p^* / q^*) \exp(-b_0 x) \tag{6}$$

and

$$w_0(x) = (r^* / q^*) \exp(-b_1 (1 - x)) \tag{7}$$

where

$$p^* = [\varphi_1 - u_0(0)] - [\varphi_2 - u_0(1)] \exp(-b_1)$$

$$q^* = 1 - \exp(-b_0 b_1)$$

and

$$r^* = [\varphi_2 - u_0(1)] - [\varphi_1 - u_0(0)] \exp(-b_0) .$$

It can be observed that [3, 11], if u is the solution of (1a,b) and U is given by (4)

$$| u(x - U)(x) | \le C \sqrt{\varepsilon} \ , \text{ for } 0 \le x \le 1 \tag{8}$$

for sufficiently smooth functions b(x) and f(x).

From (4), the left terminal boundary condition is taken as

$$u(x_p) = u_0(x_p) + v_0(x_p) + w_0(x_p) = \varphi_3 \ . \tag{9}$$

Note that the left terminal point x_p will be of the form

$$x_p = t_p \sqrt{\varepsilon} \text{ where } t_p = 1, 10, 20, 30, \dots .$$

Right terminal boundary condition

Let x_q be the right terminal boundary point or width or thickness of the right boundary layer region. Using the zeroth order asymptotic expansion for the solution of the SPP(1a,b), the right terminal boundary condition is taken as

$$u(x_q) = u_0(x_q) + v_0(x_q) + w_0(x_q) = \varphi_4 \ . \tag{10}$$

Note that the right terminal point x_q will be of the form

$$x_q = 1 - t_p \sqrt{\varepsilon}$$

where $t_p = 1, 10, 20, 30, \dots$. and $x_q = 1 - x_p$.

Now using the left and right terminal boundary conditions we split the SPP(1a,b) into three two point boundary value problems(TPBVP.

Left boundary layer region problem

The left terminal point x_p is common to both the left boundary layer region and the outer region. We have the left boundary layer region problem as a TPBVP as follows:

$$-\varepsilon\, u''(x) + b(x)\, u(x) = f(x), \tag{11a}$$

for $0 < x < x_p$ with

$$u(0) = \varphi_1 \text{ and } u(x_p) = \varphi_3. \tag{11b}$$

Right boundary layer region problem

The right terminal point x_q is common to both the right boundary layer region and the outer region. We have the right boundary layer region problem as a TPBVP as follows:

$$-\varepsilon\, u''(x) + b(x)\, u(x) = f(x), \tag{12a}$$

for $x_q < x < 1$ with

$$u(x_q) = \varphi_4 \text{ and } u(1) = \varphi_2 \tag{12b}$$

Outer region problem

Using the left terminal point x_p and the right terminal point x_q we have the outer region problem as a TPBVP as follows:

$$-\varepsilon\, u''(x) + b(x)\, u(x) = f(x), \tag{13a}$$

for $x_p < x < x_q$ with

$$u(x_p) = \varphi_3 \text{ and } u(x_q) = \varphi_4. \tag{13b}$$

Solution of the original problem

After solving the above three problems, we combine the solutions of these three problems to obtain an approximate solution to the original problem over the interval [0, 1].

We repeat the process for various choices of terminal points x_p and x_q until the solution profiles do not differ much from iteration to iteration. For a computational point of view, we use error estimates of the form

$$\left| U(x)^{m+1} - U(x)^{m} \right| \le \delta,\ 0 < x < x_p \tag{14a}$$

and

$$\left| U(x)^{n+1} - U(x)^{n} \right| \le \xi,\ x_q < x < 1 \tag{14b}$$

where $U(x)^m$ and $U(x)^n$ are m^{th} and n^{th} iterations of the left and right boundary layer region solutions respectively and δ and ξ are prescribed tolerance bounds.

Numerical method

We use an uniform and optimal finite difference scheme for the numerical solution of the SPP(1a,b) to solve the above three TPBVPs which is presented in [3, 15] and it is defined as follows:

$$L^h u_i \equiv -\varepsilon\, \sigma(\rho)\, D_+ D_- u_i + b(x_i)\, u_i = f(x_i),\ 0 < I < N-1, \tag{15a}$$

$u_0 = \varphi_1$ and $u_N = \varphi_2$ (15b)

where

$$\sigma(\rho) = \sigma(\rho\sqrt{b(x_i)})\,\sigma(-\rho\sqrt{b(x_i)}),\ \rho = h/\sqrt{\varepsilon}\ . \qquad (15c)$$

It is proved that in [15],

$$|u(x_i) - u_i| \le C \min(c\,h, \sqrt{\varepsilon})$$

where u(x) and u_i are the solutions of SPP(1a.b) and (15a-c) respectively.

Error estimates-regions wise

Using maximum principle we derive error estimates for the solutions of the left boundary layer region problem, right boundary layer region problem and outer region problem in Theorem 2, 3 and 4 respectively. The maximum principle is stated as follows in Theorem 1 [3, 15]:

Theorem 1.

Let v be any smooth function and L be the operator defined as in (1a).

i. if $v(0) \ge 0$, $v(1) \ge 0$, and $Lv(x) \ge 0$, for $x \in (0, 1)$, then we have $v(x) \ge 0$, for all $x \in [0, 1]$,
ii. for all $x \in [0, 1]$, we have $|v(x)| \le C \max(|v(0)|, |v(1)|, \max|Lv(y)|)$, $y \in [0, 1]$ and $C > 0$.

Proof : See Doolan et al., [3, 15] for proof of Theorem 1.

Theorem 2.

Let u and u^1 be the solutions of the SPP(1a,b) and (11a,b) respectively. Then , for all $x \in [0, x_p,]$,

$$|u(x) - u^1(x)| \le C\sqrt{\varepsilon} \qquad (16)$$

where C is independent of □ .

Proof. For all $0 < x < x_p$, we have

$$L[u(x) - u^1(x)] = L\,u(x) - L\,u^1(x) = f(x) - f(x) = 0.$$

For x=0, $u(0) - u^1(0) = \varphi_1 - \varphi_1 = 0$.

And for $x = x_p$, $u(x_p) - u^1(x_p,) = u(x_p) - \varphi_3$

$= u(x_p) - [u(x_p) + O(\sqrt{\varepsilon})]$

$= O(\sqrt{\varepsilon})$.

Using maximum principle, for all $x \in [0, x_p,]$, we have

$$|u(x) - u^1(x)| \le |u(x_p) - u^1(x_p,)| \le C\sqrt{\varepsilon}\ .$$

Theorem 3.
Let u and u^3 be the solutions of the SPP(1a,b) and (12a,b) respectively. Then , for all $x \in [x_q, 1]$,

$$|u(x) - u^3(x)| \leq C\sqrt{\varepsilon} \quad (17)$$

where C is independent of □ .

Proof. For all $x_q < x < 1$, we have

$$L[u(x) - u^3(x)] = L\,u(x) - L\,u^3(x) = f(x) - f(x) = 0.$$

For x=1, $u(1) - u^3(1) = -\varphi_2 - \varphi_2 = 0$.

And for $x = x_q$, $u(x_q) - u^3(x_q,) = u(x_q) - \varphi_4$

$= u(x_q) - [\,u(x_q) + O(\sqrt{\varepsilon})\,]$

$= O(\sqrt{\varepsilon})$.

Using maximum principle, for all $x \in [x_q, 1]$, we have

$|u(x) - u^3(x)| \leq |u(x_q) - u^3(x_q,)| \leq C\sqrt{\varepsilon}$.

Theorem 4.
Let u and u^2 be the solutions of the SPP(1a,b) and (13a,b) respectively. Then , for all $x \in [x_p, x_q]$,

$$|u(x) - u^2(x)| \leq C\sqrt{\varepsilon} \quad (18)$$

where C is independent of □ .

Proof. For all $x_p < x < x_q$, we have

$$L[u(x) - u^2(x)] = L\,u(x) - L\,u^2(x) = f(x) - f(x) = 0.$$

For $x = x_p$, $u(x_p) - u^2(x_p) = u(x_p) - \varphi_3$

$= u(x_p) - [\,u(x_p) + O(\sqrt{\varepsilon})\,]$

$= O(\sqrt{\varepsilon})$.

And for $x = x_q$, $u(x_q) - u^2(x_q,) = u(x_q) - \varphi_4$

$= u(x_q) - [\,u(x_q) + O(\sqrt{\varepsilon})\,]$

$= O(\sqrt{\varepsilon})$.

Using maximum principle, for all $x \in [x_p, x_q]$, we have

$|u(x) - u^2(x)| \leq \max(\,|u(x_p) - u^2(x_p)|\,,\,|u(x_q) - u^2(x_q,)|\,)$

$\leq C\sqrt{\varepsilon}$.

Error estimates-Numerical solutions

Using the discrete maximum principle we derive error estimates for the numerical solutions of the left boundary layer region problem, right boundary layer region problem and outer region problem in Theorem 6, 7 and 8 respectively using the numerical method (15a-c). The discrete maximum principle is stated as follows in Theorem 5 [3, 15]:

Theorem 5.

Let v_i be a mesh function and L^h be the operator defined as in (15a).

i. if $v_0 \geq 0$, $v_N \geq 0$, and $L^h v_i \geq 0$, for all $1 \leq i \leq N-1$, then we have $v_i \geq 0$, for all $0 \leq i \leq N$,

ii. for all $0 \leq i \leq N$, we have $| v(x) | \leq C \max(|v_0|, |v_N|, \max |L^h v_j|)$, for all $0 \leq j \leq N$ and $C > 0$.

Proof: See Doolan et al., [3, 15] for proof of Theorem 5.

Theorem 6.

Let u^1 and u^1_i be the solution of the TPBVP (11a,b) and the numerical solution of the TPBVP (11a,b) using the scheme(15a -c) respectively. Then , for all x $\in [0, x_p,]$, $0 \leq i \leq N$

$$| u^1(x_i) - u^1_i | \leq C \min(h_1 , \sqrt{\varepsilon}) \tag{19}$$

where C is independent of i, h_1 and □ .

Proof. See [3, 15].

Theorem 7.

Let u^3 and u^3_i be the solution of the TPBVP(12a,b) and the numerical solution of the TPBVP (12a,b) using the scheme (15a-c) respectively. Then , for all $x \in [x_q , 1]$, $0 \leq i \leq N$

$$|u^3(x_i) - u^3_i| \leq C \min(h_1, \sqrt{\varepsilon}) \tag{20}$$

where C is independent of i, h_1 and □ .

Proof:. See [3, 15].

Theorem 8.

Let u^2 and u^2_i be the solution of the TPBVP(13a,b) and the numerical solution of the TPBVP (13a,b) using the scheme(15a-c) respectively. Then , for all

$x \in [x_p, x_q]$, $0 \le i \le N$

$$\left| u^2(x_i) - u^2{}_i \right| \le C \min(h_2, \sqrt{\varepsilon}) \tag{21}$$

where C is independent of i, h_2 and □ .

Proof. See [3, 15].

Error Estimate-Computational Method

Using the maximum principle, we derive error estimates between the solution of the original SPP(1a,b) and the numerical solutions of the left boundary layer region problem, right boundary layer region problem and outer region problem in Theorem 9, 10 and 11 respectively.

Theorem 9.

Let u and $u^1{}_i$ be the solution of the SPP (1a,b) and the numerical solution of the problem(11a,b) using the scheme(15a-c) respectively. Then , for all $x \in [0, x_p,]$ and for $0 \le i \le N$

$$\left| u(x_i) - u^1{}_i \right| \le C(\sqrt{\varepsilon} + \min(h_1, \sqrt{\varepsilon})) \tag{22}$$

where C is independent of i, h_1 and □ .

Proof. Using triangle inequality, we have

$$\left| u(x_i) - u^1{}_i \right| \le \left| u(x_i) - u^1((x_i) \right| + \left| u^1(x_i) - u^1{}_i \right|.$$

From the estimate(16) and (19)

$$\left| u(x_i) - u^1{}_i \right| \le C(\sqrt{\varepsilon} + \min(h_1, \sqrt{\varepsilon})).$$

Theorem 10..

Let u and $u^3{}_i$ be the solution of the SPP(1a,b) and the numerical solution of the problem(12a,b) using the scheme (15a-c) respectively . Then , for all $x \in [x_q, 1]$, and for $0 \le i \le N$

$$\left| u(x_i) - u^3{}_i \right| \le C(\sqrt{\varepsilon} + \min(h_1, \sqrt{\varepsilon})) \tag{23}$$

where C is independent of i, h_1 and □ .

Proof:. Using triangle inequality, we have

$$\left| u(x_i) - u^3{}_i \right| \le \left| u(x_i) - u^3((x_i) \right| + \left| u^3(x_i) - u^3{}_i \right|.$$

From the estimate(17) and (20)

$$\left| u(x_i) - u^3{}_i \right| \le C(\sqrt{\varepsilon} + \min(h_1, \sqrt{\varepsilon})) .$$

Theorem 11.
Let u and $u^2{}_i$ be the solutions of the SPP(1a,b) and the numerical solution of the problem(13a,b) using the scheme(15a-c) respectively . Then , for all x $\in [x_p, x_q]$, , and for $0 \le i \le N$

$$\left| u(x_i) - u^2{}_i \right| \le C(\sqrt{\varepsilon} + \min(h_2, \sqrt{\varepsilon})) \quad (24)$$

where C is independent of i, h_2 and □ .

Proof. Using triangle inequality, we have

$$\left| u(x_i) - u^2{}_i \right| \le \left| u(x_i) - u^2((x_i) \right| + \left| u^2(x_i) - u^2{}_i \right| .$$

From the estimate(18) and (21)

$$\left| u(x_i) - u^2{}_i \right| \le C(\sqrt{\varepsilon} + \min(h_2, \sqrt{\varepsilon})) .$$

Remark. It is to be noted that the mesh size h_1 used in both the left and right boundary layer regions are same. And the mesh size h_2 used in the outer layer region is not equal to h_1 .

Numerical experiment
To demonstrate the applicability of the computational method , we have implemented it on two SPPs. Computed results are tabulated in Tables. From the Tables, the underlined value indicates that it is a terminal boundary conditions obtained from (9) and (10) and the corresponding x value denotes terminal points x_p and x_q respectively.

In the last column of the Tables , we have given the absolute error of the numerical solution at $x_p = 30\sqrt{\varepsilon}$ and $x_q = 1 - 30\sqrt{\varepsilon}$ to the exact solution. The mesh size used in Tables 1A and 2A are $h_2 = 0.1$ and $h_1 = 10^{-4}, 10^{-3}, 2\times10^{-3}, 3\times10^{-3}$, … for the intervals got for t_p = 1, 10, 20, 30, … respectively. The mesh size used in Tables 1B and 2B are $h_2 = 0.1$ and $h_1 = 10^{-5}, 10^{-4}, 2\times10^{-4}, 3\times10^{-4}$, … for the intervals got for t_p = 1, 10, 20, 30, … respectively.

Example 1.
Consider the following homogeneous SPP from Bender[1]

- □ $u''(x) + u(x) = 0$, $0 < x < 1$,

$u(0) = 1$, $u(1) = 1$.

The numerical results are presented in Tables 1A and 1B, for $\varepsilon = 10^{-6}$ and 10^{-8} .

Example 2.
Consider the following non-homogeneous SPP from Doolan et. al.[3]

$$-\varepsilon u''(x) + u(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x),\ 0 < x < 1,$$
$$u(0) = 0,\ u(1) = 0.$$

The numerical results are presented in Tables 2A and 2B, for $\varepsilon = 10^{-6}$ and 10^{-8}.

Discussion and conclusion

We have presented a practical method, exactly implemented on a computer to solve singularly perturbed two point boundary value problems without a fist derivative term. We have demonstrated that the computational method approximates the exact solution well, with two examples.

The present method gives more mesh points inside the boundary layers with an a priori chosen accuracy, even though this method is not uniformly convergent. Uniform and optimal schemes with constant mesh become impractical if one wants to find the local behavior of the solution in the neighborhood of ε when ε is small. The present method is practical in such situation.

It can be observed from Tables that the present method approximates the exact solution very well.

All computations were performed in Pascal single precision on a Micro Vax II computer at Bharathidasan University, Tiruchirapalli-620 024, Tamil Nadu, India.

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Table 1A : Numerical results for example 1 ε = 1.0000E-06

t_p x	1 u(x)	10 u(x)	20 u(x)	30 u(x)	EXACT SOLUTION	ERROR AT t_p = 30
0.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00
1.00000E-04	9.04837E-01	9.04837E-01	9.04837E-01	9.04837E-01	9.04837E-01	2.98023E-07
2.00000E-04	8.18730E-01	8.18730E-01	8.18730E-01	8.18730E-01	8.18731E-01	6.55651E-07
3.00000E-04	7.40818E-01	7.40817E-01	7.40817E-01	7.40817E-01	7.40818E-01	9.53674E-07
7.00000E-04	4.96585E-01	4.96584E-01	4.96584E-01	4.96584E-01	4.96585E-01	1.69873E-06
8.00000E-04	4.49329E-01	4.49327E-01	4.49327E-01	4.49327E-01	4.49329E-01	1.75834E-06
9.00000E-04	4.06569E-01	4.06568E-01	4.06568E-01	4.06568E-01	4.06570E-01	1.81794E-06
1.00000E-03	3.67879E-01	3.67878E-01	3.67878E-01	3.67878E-01	3.67879E-01	1.81794E-06
2.00000E-03		1.35334E-01	1.35334E-01	1.35334E-01	1.35335E-01	1.32620E-06
4.00000E-03		1.83153E-02	1.83153E-02	1.83153E-02	1.83156E-02	3.55765E-07
6.00000E-03		2.47868E-03	2.47868E-03	2.47868E-03	2.47875E-03	7.07805E-08
8.00000E-03		3.35449E-04	3.35449E-04	3.35449E-04	3.35463E-04	1.42318E-08
1.00000E-02		4.53999E-05	4.53974E-05	4.53974E-05	4.54000E-05	2.56841E-09
1.20000E-02			6.14378E-06	6.14378E-06	6.14421E-06	4.29736E-10
1.40000E-02			8.31459E-07	8.31459E-07	8.31530E-07	7.04858E-11
1.60000E-02			1.12524E-07	1.12524E-07	1.12535E-07	1.12195E-11
1.80000E-02			1.52283E-08	1.52283E-08	1.52300E-08	1.73195E-12
2.00000E-02			2.06115E-09	2.06089E-09	2.06116E-09	2.63345E-13
2.20000E-02				2.78908E-10	2.78947E-10	3.95517E-14
2.40000E-02				3.77455E-11	3.77513E-11	5.79745E-15
2.60000E-02				5.10823E-12	5.10910E-12	8.64326E-16
2.80000E-02				6.91317E-13	6.91441E-13	1.24250E-16
3.00000E-02				9.35762E-14	9.35764E-14	1.76183E-19
4.00000E-02					4.24837E-18	----------
1.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
2.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
3.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
4.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
5.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
6.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
7.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
8.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
9.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
9.60000E-01					4.24827E-18	--------
9.70000E-01				9.35791E-14	9.35791E-14	0.00000E+00
9.72000E-01				6.91318E-13	6.91443E-13	1.25171E-16
9.74000E-01				5.10823E-12	5.10898E-12	7.47232E-16
9.76000E-01				3.77455E-11	3.77518E-11	6.22766E-15
9.78000E-01				2.78908E-10	2.78943E-10	3.53051E-14
9.80000E-01			2.06119E-09	2.06089E-09	2.06119E-09	2.98872E-13
9.82000E-01			1.52283E-08	1.52283E-08	1.52299E-08	1.61471E-12
9.84000E-01			1.12524E-07	1.12524E-07	1.12538E-07	1.41185E-11
9.86000E-01			8.31459E-07	8.31459E-07	8.31530E-07	7.12816E-11
9.88000E-01			6.14378E-06	6.14378E-06	6.14407E-06	2.83308E-10
9.90000E-01		4.54004E-05	4.53974E-05	4.53974E-05	4.54004E-05	2.95768E-09
9.92000E-01		3.35449E-04	3.35449E-04	3.35449E-04	3.35457E-04	8.46921E-09
9.94000E-01		2.47868E-03	2.47868E-03	2.47868E-03	2.47880E-03	1.15484E-07
9.96000E-01		1.83153E-02	1.83153E-02	1.83153E-02	1.83155E-02	1.99303E-07
9.98000E-01		1.35334E-01	1.35334E-01	1.35334E-01	1.35339E-01	4.79817E-06
9.99000E-01	3.67879E-01	3.67878E-01	3.67878E-01	3.67878E-01	3.67884E-01	6.52671E-06
9.99100E-01	3.67884E-01	4.06568E-01	4.06568E-01	4.06568E-01	4.06582E-01	1.37985E-05
9.99200E-01	4.15859E-01	4.49327E-01	4.49327E-01	4.49327E-01	4.49323E-01	4.32134E-06
9.99500E-01	5.86892E-01	6.06529E-01	6.06529E-01	6.06529E-01	6.06516E-01	1.27554E-05
9.99600E-01	6.54840E-01	6.70319E-01	6.70319E-01	6.70319E-01	6.70316E-01	3.33786E-06
1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00

Table 1B : Numerical results for example 1 ε = 1.0000E-08

t_p x	1 u(x)	10 u(x)	20 u(x)	30 u(x)	EXACT SOLUTION	ERROR AT t_p = 30
0.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00
1.00000E-05	9.04837E-01	9.04837E-01	9.04837E-01	9.04837E-01	9.04837E-01	2.98023E-07
2.00000E-05	8.18730E-01	8.18730E-01	8.18730E-01	8.18730E-01	8.18731E-01	6.55651E-07
3.00000E-05	7.40818E-01	7.40817E-01	7.40817E-01	7.40817E-01	7.40818E-01	8.94070E-07
7.00000E-05	4.96585E-01	4.96584E-01	4.96584E-01	4.96584E-01	4.96585E-01	1.63913E-06
8.00000E-05	4.49329E-01	4.49327E-01	4.49327E-01	4.49327E-01	4.49329E-01	1.72853E-06
9.00000E-05	4.06569E-01	4.06568E-01	4.06568E-01	4.06568E-01	4.06570E-01	1.78814E-06
1.00000E-04	3.67879E-01	3.67878E-01	3.67878E-01	3.67878E-01	3.67879E-01	1.78814E-06
2.00000E-04		1.35334E-01	1.35334E-01	1.35334E-01	1.35335E-01	1.29640E-06
4.00000E-04		1.83153E-02	1.83153E-02	1.83153E-02	1.83156E-02	3.48315E-07
6.00000E-04		2.47859E-03	2.47868E-03	2.47868E-03	2.47875E-03	7.07805E-08
8.00000E-04		3.34735E-04	3.35449E-04	3.35449E-04	3.35463E-04	1.39116E-08
1.00000E-03		4.53999E-05	4.53974E-05	4.53974E-05	4.53999E-05	2.52476E-09
1.20000E-03			6.14378E-06	6.14378E-06	6.14421E-06	4.29736E-10
1.40000E-03			8.31458E-07	8.31459E-07	8.31529E-07	6.96900E-11
1.60000E-03			1.12520E-07	1.12524E-07	1.12535E-07	1.10063E-11
1.80000E-03			1.51959E-08	1.52283E-08	1.52300E-08	1.70175E-12
2.00000E-03			2.06115E-09	2.06089E-09	2.06115E-09	2.59570E-13
2.20000E-03				2.78908E-10	2.78947E-10	3.90243E-14
2.40000E-03				3.77455E-11	3.77513E-11	5.79745E-15
2.60000E-03				5.10823E-12	5.10909E-12	8.54351E-16
2.80000E-03				6.91317E-13	6.91440E-13	1.23003E-16
3.00000E-03				9.35761E-14	9.35761E-14	0.00000E+00
4.00000E-03					4.24835E-18	--------
1.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
2.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
3.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
4.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
5.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
6.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
7.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
8.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
9.00000E-01	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
9.96000E-01					4.24800E-18	--------
9.97000E-01				9.35566E-14	9.35566E-14	0.00000E+00
9.97200E-01				6.91314E-13	6.91524E-13	2.10064E-16
9.97400E-01				5.10823E-12	5.10836E-12	1.24466E-16
9.97600E-01				3.77455E-11	3.77585E-11	1.29237E-14
9.97800E-01				2.78908E-10	2.78926E-10	1.82632E-14
9.98000E-01			2.06168E-09	2.06089E-09	2.06168E-09	7.90257E-13
9.98200E-01			1.51960E-08	1.52283E-08	1.52299E-08	1.58629E-12
9.98400E-01			1.12520E-07	1.12524E-07	1.12505E-07	1.94689E-11
9.98600E-01			8.31458E-07	8.31459E-07	8.31579E-07	1.20451E-10
9.98800E-01			6.14378E-06	6.14378E-06	6.14296E-06	8.18090E-10
9.99000E-01		4.54058E-05	4.53974E-05	4.53974E-05	4.54058E-05	8.37099E-09
9.99200E-01		3.34736E-04	3.35449E-04	3.35449E-04	3.35417E-04	3.15194E-08
9.99400E-01		2.47859E-03	2.47868E-03	2.47868E-03	2.47924E-03	5.57629E-07
9.99600E-01		1.83153E-02	1.83153E-02	1.83153E-02	1.83144E-02	8.92207E-07
9.99800E-01		1.35334E-01	1.35334E-01	1.35334E-01	1.35371E-01	3.70592E-05
9.99900E-01	3.67818E-01	3.67878E-01	3.67878E-01	3.67878E-01	3.67818E-01	5.92470E-05
9.99910E-01	4.06516E-01	4.06568E-01	4.06568E-01	4.06568E-01	4.06557E-01	1.04606E-05
9.99920E-01	4.49282E-01	4.49327E-01	4.49327E-01	4.49327E-01	4.49376E-01	4.92036E-05
9.99950E-01	6.06503E-01	6.06529E-01	6.06529E-01	6.06529E-01	6.06480E-01	4.88758E-05
9.99960E-01	6.70298E-01	6.70319E-01	6.70319E-01	6.70319E-01	6.70355E-01	3.65973E-05
1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	1.00000E+00	0.00000E+00

Table 2A : Numerical results for example 2 ε = 1.0000E-06

t_p x	1 u(x)	10 u(x)	20 u(x)	30 u(x)	EXACT SOLUTION	ERROR AT t_p = 30
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-04	-9.51623E-02	-9.51620E-02	-9.51620E-02	-9.51620E-02	-9.51625E-02	4.69387E-07
2.00000E-04	-1.81269E-01	-1.81268E-01	-1.81268E-01	-1.81268E-01	-1.81269E-01	1.02818E-06
3.00000E-04	-2.59180E-01	-2.59179E-01	-2.59179E-01	-2.59179E-01	-2.59181E-01	1.51992E-06
7.00000E-04	-5.03409E-01	-5.03407E-01	-5.03407E-01	-5.03407E-01	-5.03410E-01	3.21865E-06
8.00000E-04	-5.50664E-01	-5.50661E-01	-5.50661E-01	-5.50661E-01	-5.50665E-01	3.63588E-06
9.00000E-04	-5.93422E-01	-5.93418E-01	-5.93418E-01	-5.93418E-01	-5.93422E-01	4.05312E-06
1.00000E-03	-6.32111E-01	-6.32106E-01	-6.32106E-01	-6.32106E-01	-6.32111E-01	4.41074E-06
2.00000E-03		-8.64618E-01	-8.64618E-01	-8.64618E-01	-8.64625E-01	7.39098E-06
4.00000E-03		-9.81517E-01	-9.81517E-01	-9.81517E-01	-9.81526E-01	9.53674E-06
6.00000E-03		-9.97156E-01	-9.97155E-01	-9.97155E-01	-9.97166E-01	1.01924E-05
8.00000E-03		-9.99021E-01	-9.99019E-01	-9.99019E-01	-9.99032E-01	1.34706E-05
1.00000E-02		-9.98967E-01	-9.98953E-01	-9.98953E-01	-9.98967E-01	1.40667E-05
1.20000E-02			-9.98558E-01	-9.98558E-01	-9.98572E-01	1.41859E-05
1.40000E-02			-9.98050E-01	-9.98050E-01	-9.98064E-01	1.41263E-05
1.60000E-02			-9.97459E-01	-9.97459E-01	-9.97473E-01	1.42455E-05
1.80000E-02			-9.96791E-01	-9.96789E-01	-9.96803E-01	1.41859E-05
2.00000E-02			-9.96054E-01	-9.96040E-01	-9.96054E-01	1.41263E-05
2.20000E-02				-9.95213E-01	-9.95227E-01	1.43051E-05
2.40000E-02				-9.94307E-01	-9.94321E-01	1.42455E-05
2.60000E-02				-9.93324E-01	-9.93338E-01	1.41263E-05
2.80000E-02				-9.92264E-01	-9.92276E-01	1.23978E-05
3.00000E-02				-9.91136E-01	-9.91136E-01	0.00000E+00
4.00000E-02					-9.84279E-01	---------
1.00000E-01	-9.04434E-01	-9.04434E-01	-9.04434E-01	-9.04434E-01	-9.04434E-01	0.00000E+00
2.00000E-01	-6.54274E-01	-6.54274E-01	-6.54274E-01	-6.54274E-01	-6.54268E-01	6.07967E-06
3.00000E-01	-3.45125E-01	-3.45125E-01	-3.45125E-01	-3.45125E-01	-3.45131E-01	6.10948E-06
4.00000E-01	-9.51784E-02	-9.51784E-02	-9.51784E-02	-9.51784E-02	-9.51944E-02	1.59964E-05
5.00000E-01	1.93554E-05	1.93554E-05	1.93554E-05	1.93554E-05	-3.99690E-07	1.97551E-05
6.00000E-01	-9.59220E-02	-9.59220E-02	-9.59220E-02	-9.59220E-02	-9.59380E-02	1.59666E-05
7.00000E-01	-3.46328E-01	-3.46328E-01	-3.46328E-01	-3.46328E-01	-3.46334E-01	6.07967E-06
8.00000E-01	-6.55476E-01	-6.55476E-01	-6.55476E-01	-6.55476E-01	-6.55470E-01	6.13928E-06
9.00000E-01	-9.05176E-01	-9.05176E-01	-9.05176E-01	-9.05176E-01	-9.05176E-01	1.19209E-07
9.60000E-01					-9.84592E-01	--------
9.70000E-01				-9.91372E-01	-9.91372E-01	0.00000E+00
9.72000E-01				-9.92481E-01	-9.92496E-01	1.45435E-05
9.74000E-01				-9.93526E-01	-9.93542E-01	1.60933E-05
9.76000E-01				-9.94494E-01	-9.94510E-01	1.59740E-05
9.78000E-01				-9.95384E-01	-9.95400E-01	1.56164E-05
9.80000E-01			-9.96211E-01	-9.96196E-01	-9.96211E-01	1.54972E-05
9.82000E-01			-9.96931E-01	-9.96929E-01	-9.96944E-01	1.51992E-05
9.84000E-01			-9.97584E-01	-9.97584E-01	-9.97599E-01	1.49608E-05
9.86000E-01			-9.98159E-01	-9.98159E-01	-9.98174E-01	1.47820E-05
9.88000E-01			-9.98651E-01	-9.98651E-01	-9.98666E-01	1.46627E-05
9.90000E-01		-9.99045E-01	-9.99031E-01	-9.99031E-01	-9.99045E-01	1.44243E-05
9.92000E-01		-9.99083E-01	-9.99081E-01	-9.99081E-01	-9.99094E-01	1.36495E-05
9.94000E-01		-9.97202E-01	-9.97201E-01	-9.97201E-01	-9.97212E-01	1.03712E-05
9.96000E-01		-9.81547E-01	-9.81547E-01	-9.81547E-01	-9.81557E-01	9.83477E-06
9.98000E-01		-8.64632E-01	-8.64632E-01	-8.64632E-01	-8.64636E-01	3.81470E-06
9.99000E-01	-6.32111E-01	-6.32113E-01	-6.32113E-01	-6.32113E-01	-6.32112E-01	8.94070E-07
9.99100E-01	-6.32112E-01	-5.93424E-01	-5.93424E-01	-5.93424E-01	-5.93416E-01	8.58307E-06
9.99200E-01	-5.84138E-01	-5.50666E-01	-5.50666E-01	-5.50666E-01	-5.50676E-01	9.11951E-06
9.99500E-01	-4.13107E-01	-3.93468E-01	-3.93468E-01	-3.93468E-01	-3.93483E-01	1.56760E-05
9.99600E-01	-3.45160E-01	-3.29679E-01	-3.29679E-01	-3.29679E-01	-3.29684E-01	5.48363E-06
1.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00

Table 2B : Numerical results for example 2 ε = 1.0000E-08

t_p / x	1 u(x)	10 u(x)	20 u(x)	30 u(x)	EXACT SOLUTION	ERROR AT t_p = 30
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
1.00000E-05	-9.51625E-02	-9.51621E-02	-9.51621E-02	-9.51621E-02	-9.51626E-02	5.06639E-07
2.00000E-05	-1.81269E-01	-1.81268E-01	-1.81268E-01	-1.81268E-01	-1.81269E-01	1.01328E-06
3.00000E-05	-2.59181E-01	-2.59180E-01	-2.59180E-01	-2.59180E-01	-2.59182E-01	1.57952E-06
7.00000E-05	-5.03414E-01	-5.03411E-01	-5.03411E-01	-5.03411E-01	-5.03415E-01	3.51667E-06
8.00000E-05	-5.50671E-01	-5.50667E-01	-5.50667E-01	-5.50667E-01	-5.50671E-01	3.75509E-06
9.00000E-05	-5.93430E-01	-5.93426E-01	-5.93426E-01	-5.93426E-01	-5.93430E-01	4.17233E-06
1.00000E-04	-6.32120E-01	-6.32116E-01	-6.32116E-01	-6.32116E-01	-6.32120E-01	4.52995E-06
2.00000E-04		-8.64657E-01	-8.64657E-01	-8.64657E-01	-8.64664E-01	7.45058E-06
4.00000E-04		-9.81673E-01	-9.81673E-01	-9.81673E-01	-9.81683E-01	9.65595E-06
6.00000E-04		-9.97508E-01	-9.97507E-01	-9.97507E-01	-9.97518E-01	1.03116E-05
8.00000E-04		-9.99647E-01	-9.99645E-01	-9.99645E-01	-9.99658E-01	1.35899E-05
1.00000E-03		-9.99945E-01	-9.99930E-01	-9.99930E-01	-9.99945E-01	1.42455E-05
1.20000E-03			-9.99965E-01	-9.99965E-01	-9.99980E-01	1.44243E-05
1.40000E-03			-9.99966E-01	-9.99966E-01	-9.99980E-01	1.42455E-05
1.60000E-03			-9.99961E-01	-9.99960E-01	-9.99975E-01	1.43647E-05
1.80000E-03			-9.99956E-01	-9.99954E-01	-9.99968E-01	1.43647E-05
2.00000E-03			-9.99961E-01	-9.99946E-01	-9.99961E-01	1.43051E-05
2.20000E-03				-9.99938E-01	-9.99952E-01	1.43051E-05
2.40000E-03				-9.99929E-01	-9.99943E-01	1.42455E-05
2.60000E-03				-9.99919E-01	-9.99933E-01	1.40071E-05
2.80000E-03				-9.99910E-01	-9.99923E-01	1.23382E-05
3.00000E-03				-9.99911E-01	-9.99911E-01	0.00000E+00
4.00000E-03					-9.99842E-01	---------
1.00000E-01	-9.04434E-01	-9.04434E-01	-9.04434E-01	-9.04434E-01	-9.04434E-01	0.00000E+00
2.00000E-01	-6.54268E-01	-6.54268E-01	-6.54268E-01	-6.54268E-01	-6.54268E-01	5.96046E-08
3.00000E-01	-3.45131E-01	-3.45131E-01	-3.45131E-01	-3.45131E-01	-3.45131E-01	5.96046E-08
4.00000E-01	-9.51942E-02	-9.51942E-02	-9.51942E-02	-9.51942E-02	-9.51944E-02	1.56462E-07
5.00000E-01	-2.02139E-07	-2.02139E-07	-2.02139E-07	-2.02139E-07	-3.99690E-07	1.97551E-07
6.00000E-01	-9.59378E-02	-9.59378E-02	-9.59378E-02	-9.59378E-02	-9.59380E-02	1.56462E-07
7.00000E-01	-3.46334E-01	-3.46334E-01	-3.46334E-01	-3.46334E-01	-3.46334E-01	5.96046E-08
8.00000E-01	-6.55470E-01	-6.55470E-01	-6.55470E-01	-6.55470E-01	-6.55470E-01	5.96046E-08
9.00000E-01	-9.05176E-01	-9.05176E-01	-9.05176E-01	-9.05176E-01	-9.05176E-01	1.19209E-07
9.96000E-01					-9.99872E-01	--------
9.97000E-01				-9.99933E-01	-9.99933E-01	0.00000E+00
9.97200E-01				-9.99931E-01	-9.99943E-01	1.25170E-05
9.97400E-01				-9.99938E-01	-9.99952E-01	1.42455E-05
9.97600E-01				-9.99946E-01	-9.99961E-01	1.43051E-05
9.97800E-01				-9.99954E-01	-9.99968E-01	1.43647E-05
9.98000E-01			-9.99975E-01	-9.99960E-01	-9.99975E-01	1.44243E-05
9.98200E-01			-9.99968E-01	-9.99966E-01	-9.99981E-01	1.43647E-05
9.98400E-01			-9.99972E-01	-9.99971E-01	-9.99986E-01	1.43647E-05
9.98600E-01			-9.99975E-01	-9.99975E-01	-9.99989E-01	1.43647E-05
9.98800E-01			-9.99973E-01	-9.99973E-01	-9.99988E-01	1.43051E-05
9.99000E-01		-9.99951E-01	-9.99937E-01	-9.99937E-01	-9.99951E-01	1.41263E-05
9.99200E-01		-9.99652E-01	-9.99649E-01	-9.99649E-01	-9.99663E-01	1.35899E-05
9.99400E-01		-9.97511E-01	-9.97511E-01	-9.97511E-01	-9.97520E-01	9.71556E-06
9.99600E-01		-9.81675E-01	-9.81675E-01	-9.81675E-01	-9.81686E-01	1.05500E-05
9.99800E-01		-8.64657E-01	-8.64657E-01	-8.64657E-01	-8.64629E-01	2.84910E-05
9.99900E-01	-6.32181E-01	-6.32116E-01	-6.32116E-01	-6.32116E-01	-6.32181E-01	6.50287E-05
9.99910E-01	-5.93483E-01	-5.93426E-01	-5.93426E-01	-5.93426E-01	-5.93442E-01	1.59144E-05
9.99920E-01	-5.50716E-01	-5.50667E-01	-5.50667E-01	-5.50667E-01	-5.50623E-01	4.43459E-05
9.99950E-01	-3.93495E-01	-3.93466E-01	-3.93466E-01	-3.93466E-01	-3.93518E-01	5.20051E-05
9.99960E-01	-3.29700E-01	-3.29678E-01	-3.29678E-01	-3.29678E-01	-3.29643E-01	3.43323E-05
1.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00